A logical approach to fuzzy truth hedges

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Hedges

In pragmatics, hedges are linguistic terms used to mitigate the impact of an utterance.

Example

Adjectives:

- They lost an *insignificant* amount of money.

Adverbs:

- Messi is a *slightly* better player than Maradona ever was.

Clauses:

- *I'm not an expert*, but you might want to try restarting your computer.

Hedges

Lakoff (1973): hedges in fuzzy logics used not only as mitigating but also strengthening modifiers.

Brown and Levinson define hedge as “a particle, word or phrase that modifies the degree of membership of a predicate or a noun phrase in a set; it says of that membership that it is partial or true only in certain respects, or that it is more true and complete than perhaps might be expected".

Hedges

Example

Adjectives:

- They lost a *terrible* amount of money. (stressing)
- They lost an *insignificant* amount of money. (depressing)

Adverbs:

- Messi is *definitely* a better player than Maradona ever was. (stressing)
- Messi is a *slightly* better player than Maradona ever was. (depressing)

Clauses:

- You might want to try restarting the computer, *I know what I'm talking about*. (stressing)
- *I'm not an expert*, but you might want to try restarting your computer. (depressing)

Hedges

Hedge clauses that directly refer to truth:

- it is very true that (stressing)
- it is slightly true that (depressing)

Hedges in Fuzzy Set Theory

Lotfi A. Zadeh, A Fuzzy Set Theoretic Interpretation of Linguistic Hedges, *Journal of Cybernetics* 2(3) 4–34, 1972.

Interpretation in Fuzzy Set Theory: functions of the interval of truth-values into itself modifying truth-degrees of propositions.

Truth-stressing hedges: subdiagonal non-decreasing functions preserving 0 and 1.

Truth-depressing hedges: superdiagonal non-decreasing functions preserving 0 and 1.

Hedges in Mathematical Fuzzy Logic (MFL)

Hájek (2001):

BL_{vt} is the expansion of BL with a new unary connective vt (very true), the axioms

(VT1) $vt\,\varphi \to \varphi$,
(VT2) $vt(\varphi \to \psi) \to (vt\,\varphi \to vt\,\psi)$,
(VT3) $vt(\varphi \vee \psi) \to (vt\,\varphi \vee vt\,\psi)$.

and the necessitation inference rule:

(NEC) from φ infer $vt\,\varphi$

BL_{vt}-algebra

An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, vt, \overline{0}, \overline{1}\rangle$ such that

(0) $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a BL-algebra,

(1) $vt(\overline{1}) = \overline{1}$,

(2) $vt(x) \leq x$,

(3) $vt(x \to y) \leq vt(x) \to vt(y)$,

(4) $vt(x \vee y) \leq vt(x) \vee vt(y)$.

BL_{vt}-algebras form a variety and are the equivalent algebraic semantics of BL_{vt}. Chain-completeness.

$\tau(\varphi) = vt(\varphi \,\&\, \varphi)$ and $\tau^n\varphi = \tau(\ldots\tau(\tau\varphi)\ldots)$.

Local deduction theorem

$T \cup \{\varphi\} \vdash_{\mathrm{BL}_{vt}} \psi$, iff, for some n, $T \vdash_{\mathrm{BL}_{vt}} \tau^n\varphi \to \psi$.

Vychodil (2006):

$\mathrm{BL}_{vt,st}$ is the expansion of BL_{vt} with a new unary connective st (slightly true), the axioms

(ST1) $\varphi \to st\,\varphi$,
(ST2) $st\,\varphi \to \neg\, vt\,\neg\varphi$,
(ST3) $vt(\varphi \to \psi) \to (st\,\varphi \to st\,\psi)$

and the necessitation inference rule:

(NEC) from φ infer $vt\,\varphi$

$\mathrm{BL}_{vt,st}$-algebras and chain-completeness.

Hedges in Mathematical Fuzzy Logic (MFL)

System I:

(ST1) $\varphi \to st\,\varphi$,

(ST4) $\neg\, st(\bar{0})$,

(ST5) $st(\varphi \to \psi) \to (st\,\varphi \to st\,\psi)$,

System II consists of the axioms (ST1), (ST4) and

(ST6) $(\varphi \to \psi) \to (st\,\varphi \to st\,\psi)$,

Both systems also have the following rule of inference:

(RN_{st}) from $\neg\varphi$ infer $\neg\, st\,\varphi$.

Again algebraization and chain-completeness for both systems.

Deduction theorems are not studied.

Problems in these approaches

- K-like axioms impose a rather unnatural restriction on hedge functions.
- standard completeness is only proved for G_{vt}.

Rasiowa-implicative Semilinear logics [Cintula-Noguera 2010]

L is a Rasiowa-implicative logic if:

$$
\begin{array}{ll}
\text{(R)} & \vdash_{\mathrm{L}} \varphi \to \varphi, \\
\text{(MP)} & \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi, \\
\text{(T)} & \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi, \\
\text{(sCng)} & \varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \to \\
& \qquad\qquad\qquad c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n) \\
& \text{for each } n\text{-ary } c \in \mathcal{L} \text{ and each } i < n, \\
\text{(W)} & \varphi \vdash_{\mathrm{L}} \psi \to \varphi.
\end{array}
$$

Every finitary Rasiowa-implicative logic is algebraizable in the sense of Blok and Pigozzi. L-algebras form a quasivariety.

Rasiowa-implicative Semilinear logics [Cintula-Noguera 2010]

Every L-algebra satisfies $x \to x = y \to y$ for any x, y, and hence the language can be expanded by $\overline{1} = p \to p$.

If $\boldsymbol{A}$ is an L-algebra, we can defined an order relation:

$$a \leq^{\boldsymbol{A}} b \text{ iff } a \to^{\boldsymbol{A}} b = \overline{1}^{\boldsymbol{A}}.$$

Semilinear logic: complete w.r.t. chains.

Core fuzzy logics

- MTL: logic of left-continuous t-norms and their residua [Esteva-Godo (2001), Jenei-Montagna (2002)]
- Core fuzzy logics: axiomatic expansions of MTL with (sCng).
- Examples: Łukasiewicz logic, product logic, Gödel logic, BL, SBL, ΠMTL, fuzzy logics with truth-constants, classical logic,...

Core fuzzy logics are Rasiowa-implicative semilinear logics. Their equivalent algebraic semantics are varieties.

Disjunctions

A (primitive or definable) binary connective $\vee$ is called a disjunction in L whenever it satisfies:

(PD)	$\varphi \vdash_{\mathrm{L}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \vee \psi$,
(PCP)	If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$.

Given a finitary inference rule $(R)\colon \Gamma \vdash \varphi$, we define $(R^{\vee})\colon \Gamma \vee p \vdash \varphi \vee p$, where p is an arbitrary propositional variable not appearing in $\Gamma \cup \{\varphi\}$.

Proposition

Let L_1 be a logic with a disjunction $\vee$ and let L_2 be an expansion of L_1 by a set of finitary rules $\mathcal{C}$. Then, $\vee$ is a disjunction in L_2 iff $(R^{\vee})$ holds in L_2 for each $(R) \in \mathcal{C}$. In particular, $\vee$ is a disjunction in any *axiomatic* expansion of L_1.

Interplay of disjunction and semilinearity

Proposition

Let L be a finitary Rasiowa-implicative logic with a binary connective $\vee$ satisfying (PD). Consider the following two properties:

$$
\begin{array}{ll}
(\mathrm{P}) & \vdash_{\mathrm{L}} (\varphi \to \psi) \vee (\psi \to \varphi), \\
(\mathrm{DMP}) & \varphi \to \psi, \varphi \vee \psi \vdash_{\mathrm{L}} \psi \text{ and } \varphi \to \psi, \psi \vee \varphi \vdash_{\mathrm{L}} \psi.
\end{array}
$$

The following are equivalent:

(i) $\vee$ is a disjunction and satisfies (P),

(ii) L is semilinear and satisfies (DMP).

Our axiomatization of logics with truth-stressing hedges

Let L be a core fuzzy logic, and consider L_s the expansion of L with a new unary connective s (for *stresser*) defined by the following additional axioms:

(VTL1) $s\,\varphi \to \varphi$,

(VTL2) $s\,\overline{1}$,

and the following additional inference rule:

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(s\,\varphi \to s\,\psi) \vee \chi$.

Our axiomatization of logics with truth-depressing hedges

Let L be a core fuzzy logic, and consider L_d the expansion of L with a new unary connective s (for *depresser*) defined by the following additional axioms:

(STL1) $\varphi \to d\,\varphi$,

(STL2) $\neg\, d\,\overline{0}$,

and the following additional inference rule:

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(d\,\varphi \to d\,\psi) \vee \chi$.

Syntactical properties

(i) $\vdash_{\text{L}_s} \neg s\,\overline{0}$,

(ii) $\varphi \to \psi \vdash_{\text{L}_s} s\,\varphi \to s\,\psi$,

(iii) $\psi \vdash_{\text{L}_s} s\,\psi$,

(iv) $s\,\varphi, \varphi \to \psi \vdash_{\text{L}_s} s\,\psi$,

(v) $\vdash_{\text{L}_s} s(\varphi \vee \psi) \leftrightarrow s\,\varphi \vee s\,\psi$,

(vi) $\vdash_{\text{L}_s} s(\varphi \wedge \psi) \leftrightarrow s\,\varphi \wedge s\,\psi$.

Semantics

Ł$_s$-algebra

$\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, s, \overline{0}, \overline{1} \rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$ such that:

(0) $\langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is an Ł-algebra,

(1) $s(\overline{1}) = \overline{1}$,

(2) $s(x) \leq x$,

(3) if $(x \rightarrow y) \vee z = \overline{1}$ then $(s(x) \rightarrow s(y)) \vee z = \overline{1}$.

Ł$_s$-algebras form a quasivariety.

Ł$_s$ is Rasiowa-implicative semilinear logic (thus, we have chain-completeness).

Remarks on our axiomatization – 1

- $\mathrm{L} + [s(\varphi \vee \psi) \leftrightarrow s\,\varphi \vee s\,\psi] + [s(\varphi \wedge \psi) \leftrightarrow s\,\varphi \wedge s\,\psi] < \mathrm{L}_s$

Example

Algebra $\boldsymbol{A}$: Non-linear G-algebra over the lattice on $\{0, a, b, c, 1\}$ where c is the atom, with $s(1) = s(b) = 1$ and $s(a) = s(c) = s(0) = 0$.
Filter: $F = \{c, a, b, 1\}$.
$\langle \boldsymbol{A}, F\rangle$ is a model of (VTL1), (VTL2), the two monotonicity axioms and *modus ponens*.
$b \to a = c \in F$ and $s(b) \to s(a) = 1 \to 0 = 0 \notin F$.

Remarks on our axiomatization – 2

- $s(\varphi \vee \psi) \leftrightarrow s\,\varphi \vee s\,\psi$
- $s(\varphi \wedge \psi) \leftrightarrow s\,\varphi \wedge s\,\psi$

are not equivalent.

Example

Over the same G-algebra as before and filter $F = \{1\}$:

- $s(a) = s(b) = s(c) = 0$. It satisfies the monotonicity for the infimum but not for the supremum since $s(a \vee b) = s(1) = 1$ and $s(a) \vee s(b) = 0 \vee 0 = 0$.
- the identity operator except for $s(c) = 0$. This mapping satisfies the monotonicity for the supremum but not for the infimum since $s(a \wedge b) = s(0) = 0$ and $s(a) \wedge s(b) = a \wedge b = c$.

Remarks on our axiomatization – 3

- $\text{L} + \psi \vdash s\,\psi < \text{L}_s$

Example

$\boldsymbol{C}$ MTL-chain defined over $C = \{0, 1, 2, 3, 4, 5\}$ with the natural order and:

&	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	2	2
3	0	1	1	1	2	3
4	0	1	2	2	4	4
5	0	1	2	3	4	5

x	$s(x)$
0	0
1	1
2	1
3	3
4	4
5	5

Filter $F = \{4, 5\}$. For every $x \in F$, $s(x) \in F$, but $3 \to 2 = 4 \in F$, while $s(3) \to s(2) = 3 \to 1 = 3 \notin F$.

- $\mathrm{L} + \varphi \to \psi \vdash s\,\varphi \to s\,\psi < \mathrm{L}_s$

Example

$\boldsymbol{A}$ L-chain with at least 3 elements. Take $0 < a < 1$ and $\boldsymbol{A}^2$ and $s(x,y) = \langle x \wedge y, x \wedge y \rangle$.
Filter: $F = \{\langle 1,1 \rangle\}$
$\langle \boldsymbol{A}, F \rangle$ is a model of (VTL1), (VTL2), $\varphi \to \psi \vdash s\,\varphi \to s\,\psi$ and *modus ponens*.
$(\langle 1,1 \rangle \to \langle 1,a \rangle) \vee \langle a,1 \rangle = \langle 1,a \rangle \vee \langle a,1 \rangle = \langle 1,1 \rangle$, while
$(s(1,1) \to s(1,a)) \vee \langle a,1 \rangle = (\langle 1,1 \rangle \to \langle a,a \rangle) \vee \langle a,1 \rangle = \langle a,a \rangle \vee \langle a,1 \rangle = \langle a,1 \rangle \neq \langle 1,1 \rangle$.

Completeness theorems – 1

Definition

Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains.

- L has the $\mathrm{S}\mathbb{K}\mathrm{C}$ if: for every $\Gamma\{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$
- L has the $\mathrm{FS}\mathbb{K}\mathrm{C}$ if: for every finite $\Gamma\{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$

If $\mathbb{K}$ is the class of all L-chains over $[0, 1]$, we write $S\mathcal{R}C$ and $\mathrm{FS}\mathcal{R}\mathrm{C}$.

Completeness theorems – 2

Theorem (CEGGMN 2009)

Let L *be a core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then:*

- L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *iff every countable* L*-chain is embeddable into some member of* $\mathbb{K}$*.*
- *If the language is finite,* L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *iff every countable* L*-chain is partially embeddable into* $\mathbb{K}$*.*

Completeness theorems – 3

Theorem (Finite strong real completeness)

Let L be a core fuzzy logic. If L has the $\mathrm{FS}\mathcal{R}\mathrm{C}$, then the logic L_s has it as well.

Theorem

Let L be a core fuzzy logic, $\mathbb{K}$ a class of L-chains, and $\mathbb{K}_s$ the class of the L_s-chains whose s-free reducts are in $\mathbb{K}$. If L has the $\mathrm{FS}\mathbb{K}\mathrm{C}$, then L_s has the $\mathrm{FS}\mathbb{K}_s\mathrm{C}$.

Proof:

- Assume that L has the $\mathrm{FS}\mathcal{R}\mathrm{C}$.
- $\boldsymbol{A}$ L_s-chain and B a finite partial subalgebra.
- By assumption, the s-free reduct of $\boldsymbol{A}$ is partially embeddable into standard L-chains, so there is $\langle[0,1],*,\Rightarrow,\wedge,\vee,0,1\rangle$ and a partial embedding f for B.
- Take any non-decreasing and subdiagonal function $s'\colon [0,1] \to [0,1]$ satisfying $s'(f(x)) = f(s(x))$ for every $x \in B$ such that $s(x) \in B$ (for instance a linear interpolant).
- $\langle[0,1],*,\Rightarrow,\wedge,\vee,s',0,1\rangle$ an L_s-chain and f a partial embedding of L_s-chains.

Theorem (Strong real completeness)

Let L be a core fuzzy logic. If L has the $\mathrm{S}\mathcal{R}\mathrm{C}$, then the logic L_s has it as well.

Theorem

Let L be a core fuzzy logic, $\mathbb{K}$ a class of completely ordered L-chains, and $\mathbb{K}_s$ the class of the L_s-chains whose s-reducts are in $\mathbb{K}$. If L has the $\mathrm{S}\mathbb{K}\mathrm{C}$, then L_s has the $\mathrm{S}\mathbb{K}_s\mathrm{C}$.

Proof:

- Assume that L has the $S\mathcal{R}C$.
- $\boldsymbol{A}$ countable L_s-chain.
- By the assumption, the s-free reduct of $\boldsymbol{A}$ is embeddable into a standard L-chain $\boldsymbol{B} = \langle [0,1], *, \Rightarrow, \wedge, \vee, 0, 1\rangle$ by f.
- Define $s' : B \to B$: for each $z \in [0,1]$,
  $s'(z) = \sup\{f(s(x)) \mid x \in A, f(x) \leq z\}$.
- s' is a non-decreasing and subdiagonal function such that $s'(f(x)) = f(s(x))$ for any $x \in A$.
- $\boldsymbol{B}$ expanded with s' is a standard L_s-chain where $\boldsymbol{A}$ is embedded.

Deduction theorems – 1

Proposition

If L is the logic of a finite BL-chain $\boldsymbol{A}$, then:

- The chains of the variety generated by $\boldsymbol{A}$ are the subalgebras of $\boldsymbol{A}$.
- Given a BL-filter F of $\boldsymbol{A}$, the congruence defined by it, $\equiv_F$, is defined by:
  $x \equiv_F y$ iff either $x = y$ or $x, y \in F$, i.e. the congruence classes are F and the singletons $\{x\}$ for any $x \notin F$.
- The set of L_s-filters of $\boldsymbol{A}$ coincides with the set of L-filters that are closed by s.

Deduction theorems – 2

Lemma

Let L be the logic of a finite BL-chain $\boldsymbol{C}$ (with n elements, k components and with m being the maximum length of an MV component). Then, in any L_s-algebra $\boldsymbol{A}$, for every principal filter L_s-filter $Fi(\bar{a})$ generated by an element $\bar{a} \in A$ there is an element $t(\bar{a}) = (s^n(\overset{k}{\ldots} s^n(\varphi^m))^m \ldots)^m$ such that $Fi(\bar{a}) = [t(\bar{a}), \overline{1}^{\boldsymbol{A}}]$.

Theorem

For every set $\Gamma \cup \{\varphi, \psi\}$ of formulae, there is a formula $t(\varphi) = (s^n(\overset{k}{\ldots} s^n(\varphi^m))^m \ldots)^m$ such that,

$$\Gamma, \varphi \vdash_{\mathrm{L}_s} \psi \text{ iff } \Gamma \vdash_{\mathrm{L}_s} t(\varphi) \to \psi.$$

Corollary

The quasivariety associated to the logic of a finite BL-chain is a variety.

Deduction theorems – 3

Add a unary connective Δ, the rule of necessitation (from φ infer $\Delta\varphi$) and the following axiom schemata:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ3) $\Delta\varphi \to \varphi$

(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

Proposition

For each set of formulae $\Sigma \cup \{\varphi, \psi\}$ holds:
$\Sigma, \varphi \vdash_{\mathrm{MTL}_\Delta} \psi$ iff $\Sigma \vdash_{\mathrm{MTL}_\Delta} \Delta\varphi \to \psi$

In logics with Δ (primitive or definable) (MON) inference rule in L_s can equivalently be replaced by the axiom

$$(\text{MON}_\Delta)\ \Delta(\varphi \to \psi) \to (s\,\varphi \to s\,\psi)$$

and hence L_s-algebras form a variety.

Deduction theorems – 5

Definition

L_{sK} is the axiomatic extension of L_s by adding the axiom $s(\varphi \to \psi) \to (s\,\varphi \to s\,\psi)$.

$$\tau\varphi = s(\varphi \,\&\, \varphi),\ \tau^n\varphi = \tau(\ldots\tau(\tau\varphi)\ldots).$$

Theorem

For every set $\Gamma \cup \{\varphi, \psi\}$ of formulae, we have $\Gamma \cup \{\varphi\} \vdash_{L_{sK}} \psi$, iff, for some n, $\Gamma \vdash_{L_{sK}} \tau^n\varphi \to \psi$.

Theorem (Félix Bou – forthcoming paper)

For every core fuzzy logic L*, the class of* L_s*-algebras is a variety, i.e. an equational class.*

Conclusions

- Linguistic hedges admit a natural interpretation in fuzzy logic as subdiagonal (superdiagonal) non-decreasing functions preserving 0 and 1.
- For any fuzzy logic L, we have introduced a natural expansion L_s (L_d) coping with all truth-stressing hedges (truth-depressing hedges).
- General methods of algebraic logic allow to show that many of their properties are straightforward (algebraizability, completeness w.r.t. chains).
- Both L_s and L_d preserve standard completeness properties from L.
- In special cases we can prove a deduction theorem.

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