A short survey on bilattices in computer science: theory and applications

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A Survey on Bilattices in Computer Science: Theory and Applications

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Olomouc, Sep 4, 2013

Outline

1. Motivation
2. Bilattices
   - Mathematics
   - Logic
3. Future work

Why bilattices? (in a nutshell)

A bilattice is a doubly-ordered set satisfying certain properties which has applications in the following lines:

- Kripke-style truth-revision theory (underlying structure)
- Modal logic (modal operators parameterised by bilattice)
- Algebraic study of bilattice-based logics
- Logic programming (many-valuedness, wfs)
- Semantics of natural language questions
- Open constraint programming
- Uncertainty modelling
- Fuzzy logic

Some applications of bilattices I

- Bilattices were proposed by [Ginsberg'88] as a uniform framework for inference in AI
- Later, [Fitting'91] considered applications to LP, to philosophical problems such as the theory of truth, and studied their relationship with a family of many-valued systems generalizing Kleene's three-valued logics
- Bilattices were also studied by [Arieli & Avron'96], both from an algebraic and from a logical point of view, and logical bilattices were used to deal with paraconsistency and non-monotonic reasoning
- [Lallouet'05] gave a framework for Bilattice-valued Constraint Programming which allows to represent incomplete/conflicting information and to combine constraints with a set of operators

Some applications of bilattices II

- [Deschrijver et al'07] used bilattices as a framework for representing uncertain and potentially conflicting information, as in L-fuzzy set theory
- [Rivieccio et al'12-13] studied algebraically bilattice-based logics and the inclusion of implications in this framework
- [Bruns & Huth'11] defined a language for policy composition based on Belnap's four-valued logic, and shown that it neatly handles common problems in policy composition
- Bilattice-based logic reasoning has produced interesting results
- Other applications include the analysis of entailment, implicature and presupposition in natural language, the semantics of natural language questions, epistemic logic, ...

Different approaches to bilattices

- From Mathematics
  - Algebraic structure and properties
  - General constructions
- From Logic
  - Semantics of proof systems
  - Paraconsistency
- From Computer Science
  - Generalized logic programming
  - Uncertainty modeling

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1 Motivation

2 Bilattices

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Definitions

- A **prebilattice** is a tuple $(B, \leq_t, \leq_k)$, such that both $(B, \leq_t)$ and $(B, \leq_k)$ are bounded lattices3
- A **bilattice** is a tuple $(B, \leq_t, \leq_k, \neg)$, such that $(B, \leq_t, \leq_k)$ is a prebilattice and $\neg$ is an involutive negation wrt $\leq_t$ on B which preserves $\leq_k$.

Notation

- t, f (resp. $\top, \bot$) are top and bottom elements wrt $\leq_t$ (resp. $\leq_k$)
- $\vee$ (disjunction) and $\wedge$ (conjunction), resp. $\otimes$ (consensus) and $\oplus$ (accept all), are join and meet wrt $\leq_t$, resp. $\leq_k$

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Lemma

- $\neg(a \wedge b) = \neg a \vee \neg b \qquad \neg(a \vee b) = \neg a \wedge \neg b$
  $\neg(a \otimes b) = \neg a \otimes \neg b \qquad \neg(a \oplus b) = \neg a \oplus \neg b$
- $\neg f = t, \quad \neg t = f, \quad \neg\bot = \bot, \quad \neg\top = \top$

Lemma

If $-$ is a conflation operator (a negation wrt $\leq_k$ on B), then

- $-(a \wedge b) = -a \wedge -b \qquad -(a \vee b) = -a \vee -b$
  $-(a \otimes b) = -a \oplus -b \qquad -(a \oplus b) = -a \otimes -b$
- $-f = f, \quad -t = t, \quad -\bot = \top, \quad -\top = \bot$

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Definitions

- A (pre)bilattice $\mathcal{B}$ satisfies the interlacing condition if and only if $\wedge, \vee, \otimes, \oplus$ are monotonic wrt both $\leq_t$ and $\leq_k$
  In such a case we say that $\mathcal{B}$ is **interlaced**
- A (pre)bilattice is **distributive** if all the (twelve) possible distributive laws concerning $\wedge, \vee, \otimes, \oplus$ hold

Theorem (Fitting)

Every distributive (pre)bilattice is interlaced

Lemma

If B is interlaced, then

$\bot \wedge \top = f, \quad \bot \vee \top = t, \quad f \otimes t = \bot, \quad f \oplus t = \top$

Definition

Given two bounded lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$, $L_1 \odot L_2$ denotes the prebilattice $(L_1 \times L_2, \leq_t, \leq_k)$ where:

- $\langle a, b\rangle \leq_t \langle c, d\rangle$ if and only if $a \leq_1 c$ and $b \geq_2 d$
- $\langle a, b\rangle \leq_k \langle c, d\rangle$ if and only if $a \leq_1 c$ and $b \leq_2 d$

If $L_1 = L_2 = L$ and $\sqcup$ and $\sqcap$ are the join and meet in L, then:

$$\begin{aligned}
(a, b) \vee (c, d) &= (a \sqcup c, b \sqcap d)\\
(a, b) \wedge (c, d) &= (a \sqcap c, b \sqcup d)\\
(a, b) \oplus (c, d) &= (a \sqcup c, b \sqcup d)\\
(a, b) \otimes (c, d) &= (a \sqcap c, b \sqcap d)\\
\neg(a, b) &= (b, a)
\end{aligned}$$

Moreover, if 1 and 0 are the top and bottom elements of L, then $\bot = (0, 0)$, $\top = (1, 1)$, $t = (1, 0)$, $f = (0, 1)$

Constructing bilattices
The product

Theorem

- *$L \odot L$ is always interlaced*
- *$L \odot L$ is distributive if so is L*
- *Every distributive bilattice is isomorphic to $L \odot L$ for some complete distributive lattice*
- *Every interlaced bilattice is isomorphic to $L \odot L$ for some complete lattice*

If $(x, y) \in L \odot L$, then x represents the information **for** *some assertion, and y is the information* **against** *it*

Constructing bilattices

The intervals

Definition

Given a complete lattice $(L, \leq_L)$, we define $(\mathcal{I}(L), \leq_t, \leq_k)$ by:

- $\mathcal{I}(L) = \{[a, b,] \mid a \leq_L b\}$
- $[a, b] \leq_t [c, d]$ if and only if $a \leq_L c$ and $b \leq_L d$
- $[a, b] \leq_k [c, d]$ if and only if $a \leq_L c$ and $b \geq_L d$

The intuition is that intervals represent uncertain measures; $\leq_t$ compares degree of truth by ‘shifting rightwards’; $\leq_k$ compares approximations by ‘interval narrowing’

Note the similarity with the product construction.

Constructing bilattices

The intervals

Let $(L, \leq_L)$ be a complete lattice with an involutive negation $-$

- A conflation operator can be defined on $L \odot L$ by $-(a, b) = (-b, -a)$
- An element $(a, b) \in L \odot L$ is **coherent** if $(a, b) \leq_k -(a, b)$

Theorem

$\mathcal{I}(L)$ is isomorphic to the substructure of the coherent elements of $L \odot L$

Abstract approach to logic

Definition

A consequence relation in a language L is a relation $\vdash$ between 2^L and L satisfying

Reflexivity $\varphi \vdash \varphi$

Monotonicity If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$

Transitivity If $\Gamma \vdash \varphi$ and $\Gamma, \varphi \vdash \psi$, then $\Gamma \cup \Gamma' \vdash \psi$

Definition

A **propositional logic** is a pair $\langle L, \vdash \rangle$ where L is a *propositional* language and $\vdash$ is a *consequence relation* for L.

Matrices and semantics

Definitions

A **matrix** for L us a triple $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ where

- $\mathcal{V}$ is the set of *truth-values*
- $\mathcal{D}$ is the set of *designated elements* of $\mathcal{V}$
- $\mathcal{O}$ are the *truth-tables* (interpretations) of the connectives in L

A matrix allows for defining the standard semantic notions of valuation and model.

$\Gamma \vdash_{\mathcal{M}} \varphi$ iff $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\varphi)$

Theorem

The relation $\vdash_{\mathcal{M}}$ is a consequence relation on L and, hence, $\langle L, \vdash_{\mathcal{M}} \rangle$ is a propositional logic (induced by $\mathcal{M}$)

Logics via bilattices

Matrices defined on bilattices generate interesting logics because:

- It is possible to incorporate information-based considerations.
- There are ways of representing different levels of inconsistency and incompleteness.

When defining a bilattice-based logic:

- The interpretations of the connectives are usually defined by the basic $\leq_t$-ordering
- The choice of the designated elements in a multiple-valued setting is usually done as a filter or, even, a prime (ultra-)filter in $\mathcal{V}$
- Dual notions for lattice filters and prime-filters are needed

Definition

Let $(B, \leq_t, \leq_k, \neg)$ be a bilattice

1. A **bifilter** of B is a nonempty subset $F \subseteq B$ such that
    1. $a \wedge b \in F$ iff $a \in F$ and $b \in F$
    2. $a \otimes b \in F$ iff $a \in F$ and $b \in F$
2. A bifilter F is **prime** if the following holds:
    1. $a \vee b \in F$ iff $a \in F$ or $b \in F$
    2. $a \oplus b \in F$ iff $a \in F$ or $b \in F$

Logical bilattices

Definition

A **logical bilattice** is a pair (B, F) where B is a bilattice and F is a prime bifilter of B

- Logical bilattices can be used for defining logics similarly to the way Boolean algebras and prime filters are used.

Theorem

Let (B, F) be a logical bilattice. There exists a unique homomorphism $h\colon B \to \mathcal{FOUR}$ such that $h(b) \in \{t, \top\}$ if and only if $b \in F$

- Every complete distributive lattice can be turned into a logical bilattice
- Every distributive bilattice can be turned into a logical bilattice

Logical bilattices

Definition

Let B be a bilattice. Consider

- $D_k(B) = \{x \mid x \geq_k t\}$ (designated values of B wrt $\leq_k$)
- $D_t(B) = \{x \mid x \geq_t \top\}$ (designated values of B wrt $\leq_t$)

$D_k(B)$ seems to be a particularly natural candidate to play the role
of the designated values of B

Lemma

- *$t, \top \in D_k(B) \cap D_t(B)$*
- *$f, \bot \notin D_k(B) \cup D_t(B)$*
- *$D_k(B) \cup D_t(B)$ is included in any bifilter*
- *If $D_k(B) = D_t(B)$, and this holds for interlaced bilattices, then this is the smaller bifilter*

Residuated bilattices

- Recently, Jansana and Rivieccio introduced a new product bilattice construction, allowing to obtain a bilattice with two residuated pairs as a certain kind of power of a biresiduated lattice
- Actually, they use a slightly more general notion of biresiduated lattice, in that the existence of a unit element for the product is not required

Definitions

- Given a biresiduated lattice $(L, \sqcap, \sqcup, \cdot, \backslash, /)$ the **product biresiduated bilattice** is defined as $L \odot L$ together with the operations $\supset, \subset$ given, for all $(a_1, a_2), (b_1, b_2) \in L \times L$, by

$$(a_1, a_2) \supset (b_1, b_2) = (a_1 \backslash b_1, b_2 \cdot a_1)$$
$$(a_1, a_2) \subset (b_1, b_2) = (a_1 / b_1, b_1 \cdot a_2)$$

- The following derived operations are crucial. For all $\alpha, \beta \in L \times L$

$$\alpha \rightarrow \beta = (\alpha \supset \beta) \wedge (\neg\alpha \subset \neg\beta)$$
$$\alpha \leftarrow \beta = \neg\alpha \rightarrow \neg\beta$$
$$\alpha * \beta = \neg(\beta \rightarrow \neg\alpha)$$

Residuated bilattices

Any product residuated bilattice contains indeed two residuated pairs

Theorem

Let $L \odot L$ be a product biresiduated bilattice. Then, for all $\alpha, \beta, \gamma \in L \times L$,

$$\alpha * \beta \leq_t \gamma \quad \textit{iff} \quad \beta \leq_t \alpha \to \gamma \quad \textit{iff} \quad \alpha \leq_t \gamma \leftarrow \beta$$

Bou and Rivieccio proved that the lattice of congruences of any interlaced bilattice $L \odot L$ is isomorphic to that of L. The same holds in this more general context.

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3 Future work

Future work

- Kind of filters that defines a congruence in a residuated multilattice
- Algebraic properties in a residuated multilattice
- Considering the residuated operations $\odot$ and $\rightarrow$ as hyperoperations, thus leading to a complete embedding of the structure into a hyperalgebraic framework.

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