Weakest link clustering and convex geometry I

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Weakest link clusters and convex geometries: an ordinal framework

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Historic Intro to weakest link clusters

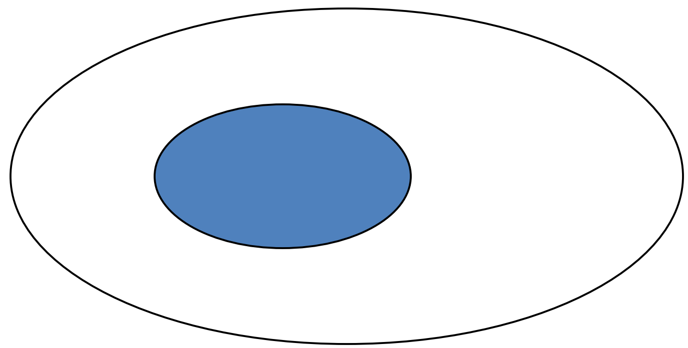
- **Joseph Mullat (1976** in Russian)
- **Ilya Muchnik and Co (1980-1990** in Russian) applied to organization structure, to consensus ordering etc.
- **Boris Mirkin, Ilya Muchnik (1997, 2002) –** some mathematics
- Similar work:
  - **Vladimir Batagelj (2002** and later – a special case, package **Pajek** for graph structure analysis)
  - others

Outline: 1st part, 2d part

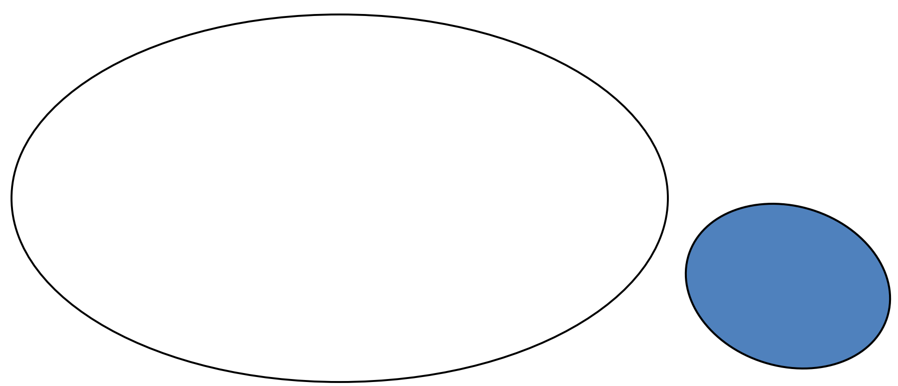
- Monotone linkage and tightness functions
- Shell layered clusters
- Serial partitioning algorithm
- Constrained shell layered clusters
- Monotone linkage and hereditary set-to-subset mapping
- Interior set system and Convex geometry
- Conclusion

Single cluster

- Core: today



- Anomalous: yesterday



Monotone linkage function $\pi(k,S)$

Inner: $k \in S$

- **in(k,S)**= $\sum_{j\in S} g_{kj}$ – incidence number [$G=(g_{kj})$ square graph incidence or rectangular context 1/0 matrix]
- **in2(k,S)**= $\sum_{j\in S} g_{kj} \sum_{t\in I} g_{jt}$ – second degree incidence number
- **c(k,S)**= $|\cup_{j\in S-k}\Gamma_j - \Gamma_k|$ - # complementary attributes/ control functions

Monotone increasing:

$$\pi(k,S) \leq \pi(k,S\cup T)$$

Monotone linkage function $\pi(k,S)$

Outer: $k \notin S$

- **L(k,S)**$=\min_{j\in S} d_{kj}$ single linkage

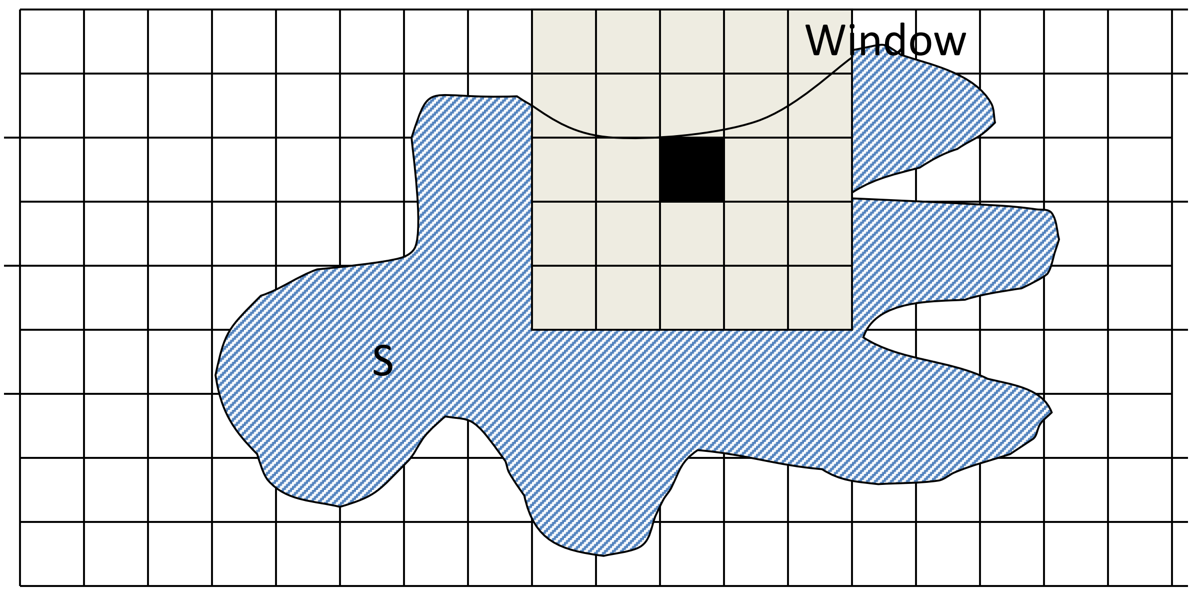
- **Ml(k,S)**$=\sum_{v\in V} \min_{j\in S}|x_{kv}-x_{jv}|/|V|$ - average minimum difference

Monotone decreasing:

$$\pi(k,S) \geq \pi(k,S\cup T)$$

Linkage function at an image

Linkage: count of pixels in $S \cap$ Window



Four monotonicity classes

Range Monotonicity	Inner	Outer
Growing	**I stick to**	
Falling		

Define **G(S)=F(I-S)**: dual inner- outer properties

Tightness function

- Given a monotone increasing **π(k,S),** its **Tightness (Weakest Link)** function

$$F_{\pi}(S) = \min_{k \in S} \pi(k,S)$$

Quasi-Convexity property

$$F_{\pi}(S \cup T) \geq \min\, [F_{\pi}(S), F_{\pi}(T)]$$

Tightness function and quasi-convexity

$$F(S \cup T) \geq \min\,[F(S), F(T)]$$

Given a quasi-convex set function **F(S)**, define

$$\pi_F(k,S) = \max_{k \in T \subseteq S} F(T)$$

which is monotone increasing

Theorem (A. Malishevski 1986, 1998).

Duality: $F_{\pi_F} = F$

Weakest link mechanism to generate all the quasi-convex functions: but HOW? No generator is known.

Weakest link core cluster

Given a monotone increasing **π(k,S),**

its **Weakest Link** cluster is **S** maximizing

$$F_\pi(S) = \min_{k \in S} \pi(k,S)$$

Not practical: Given a quasi-convex **F(S),** its maximizer is a Weakest Link cluster

Define $P_A(S)=1$ if $S=A$, $P_A(S)=0$, otherwise:

$P_A(S \cup T) \geq \min [P_A(S), P_A(T)]$. Yet HOW can one find the maximizer of P_A without enumerating all?

Shelled core cluster

- **F-Pattern S definition:**

  $$F(S) > F(S') \text{ if } S' \cap I\text{-}S \neq \emptyset$$

- **Patterns are chain nested**

- **If F is a tightness function, its minimum pattern is the largest maximizer of F**

Seriation algorithm for pattern finding I

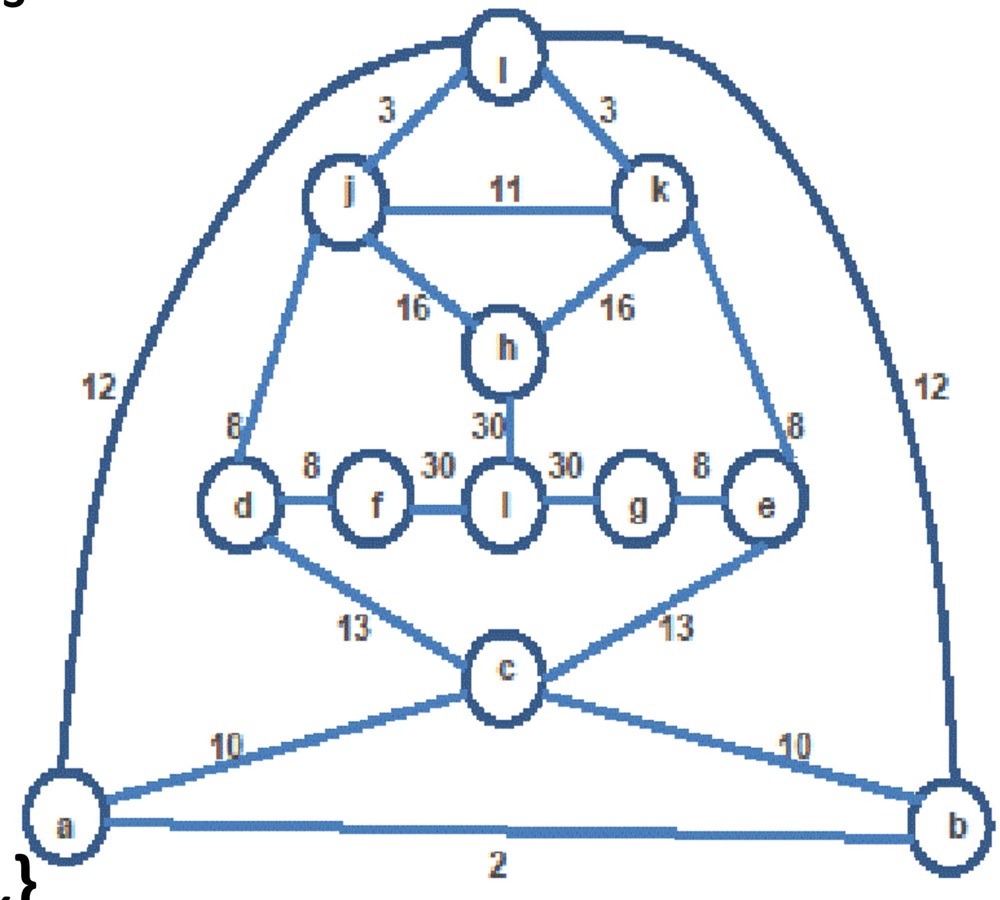
$M(S)=\{k: \pi(k,S)=\min_{j\in S} \pi(j,S)\}$

$T:=0;\ M:=\varnothing$

1. $T:=T+1$, $I:=I-M$

2. $S_T=M(I)$;
$M:=M(I);\ I:=I-M$

3. If $I=\varnothing$, end;
else go to 1



Output:

series $SS=\{S_1, S_2, \ldots, S_K\}$

PATTERN = Final_SS_Fragment_At_Max_π_Value

Seriation algorithm for pattern finding II

$\pi(k,S)=\sum_{j\in S} A(k,j)$

$S_1=M(l_1)=\{a,b\}$ at $\pi=24$

$S_2=M(l_2)=\{l\}$ at $\pi=6$

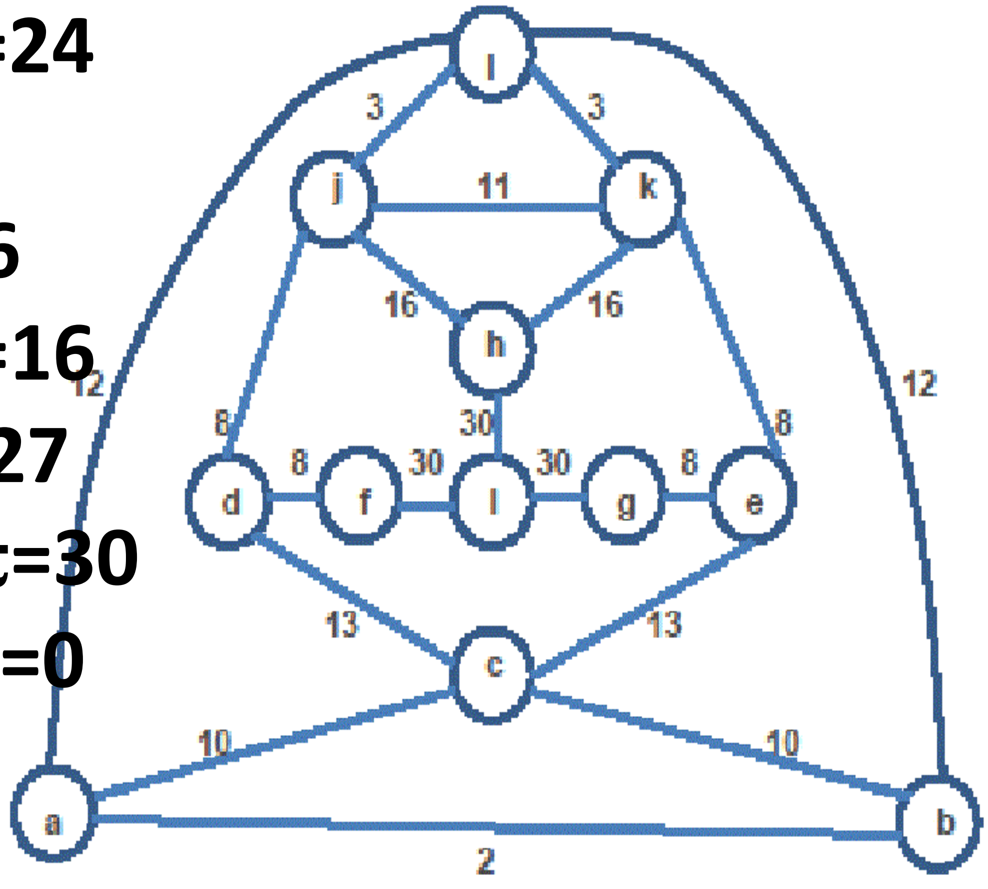
$S_3=M(l_3)=\{c\}$ at $\pi=26$

$S_4=M(l_4)=\{d,e\}$ at $\pi=16$

$S_5=M(l_5)=\{j,k\}$ at $\pi=27$

$S_6=M(l_6)=\{f,g,h\}$ at $\pi=30$

$S_7=M(l_7=\{l\})=\{i\}$ at $\pi=0$



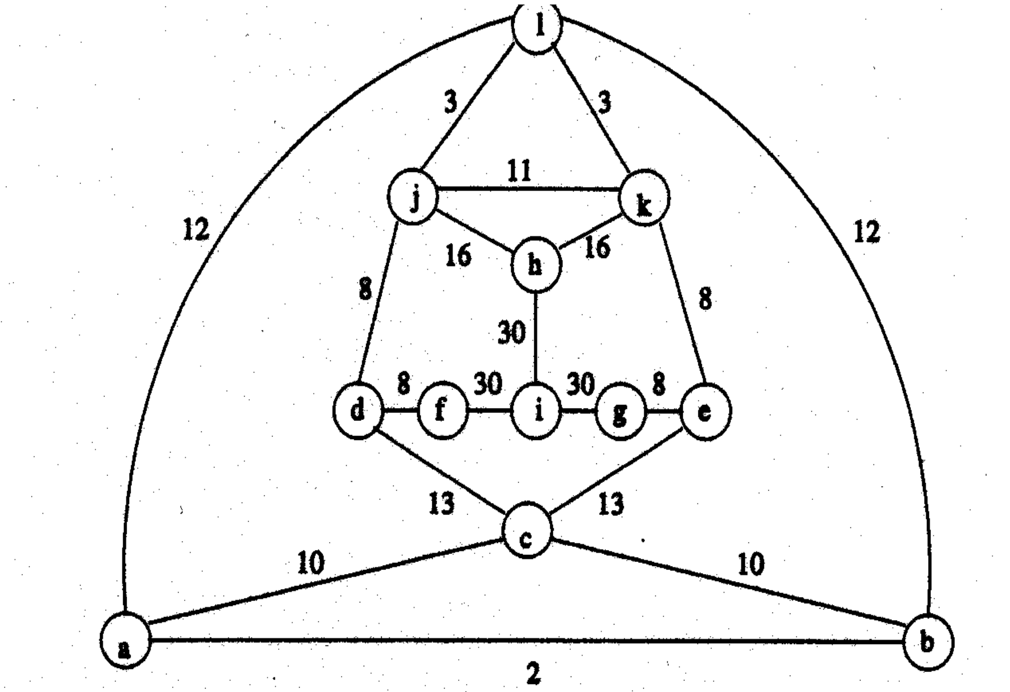
Shelled core cluster

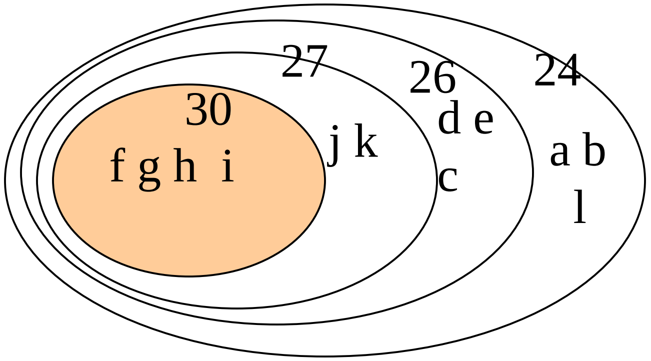
P1=fghi ^{30};   P2= fghijk ^{27};   P3=fghijkdec ^{26};   P4=I ^{24}

Structural model: shelled core

- Shelled core:

$(ab)^{24}(l)^{6}(c)\ ^{26}(de)^{16}(jk)^{27}(fgh)^{30}(i)^{0}$





Added constraints brought in by external forces

- Monotone linkage function $\pi(k,S)$
- Monotone external force linkage function $d(k,S)$

❖ Constraint $d(k,S) \leq u$ ("oversensitive" are not feasible) brings forth mapping

$$\psi_u(S)=\{\ k:\ d(k,S)\leq u\}$$ (assume not empty!)

❖ Tightness constrained:

$$F_{\pi u}(S)= \min_{k\in\psi u(S)} \pi(k,S)$$

Added constraints brought in by external forces

- Monotone linkage function **π(k,S)**
- Monotone external force linkage function **d(k,S)**

❖ $\varphi_u(S)=\{ k: d(k,S)\leq u\}$ **(assume not empty!)**

❖ **Tightness constrained:**

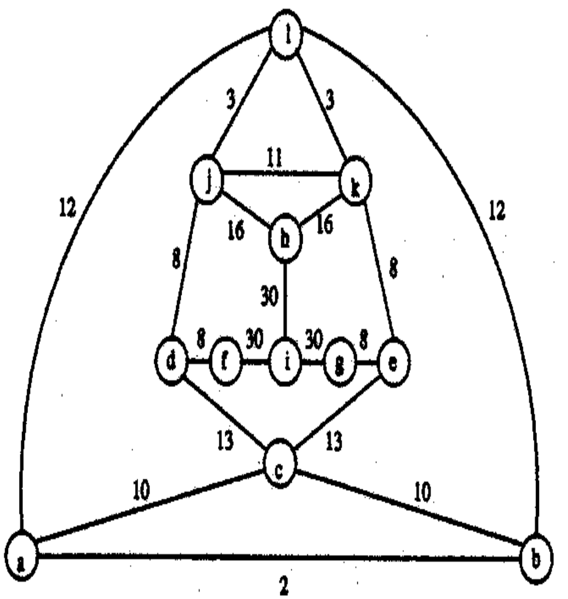
$$F_{\pi u}(S)= \min_{k\in\varphi u(S)} \pi(k,S)$$

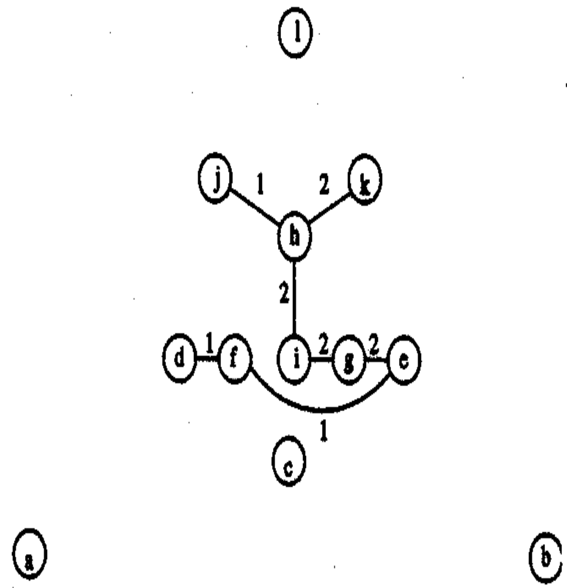
❖ Change **π(k,S):**

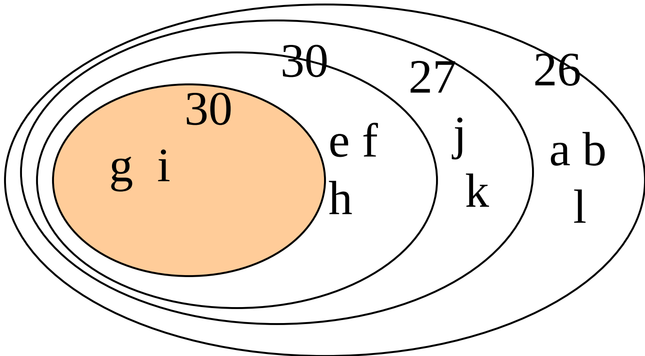
$$\pi'(k,S)=\begin{cases}\pi(k,S) & if \quad d(k,S)\leq u \\ \pi(k,S)+C & if \quad d(k,S)>u\end{cases}$$

❖ $F_{\pi u}(S) = F_{\pi'}(S)$

Added constraints brought in by external forces: $d(k,S)=\sum_{j\in S} h_{kj} \le 2$







- Shelled core under the constraint

Conclusion 1
Concepts considered

- Monotone linkage function: $I \times 2^I \Rightarrow R$
- Tightness function/Weakest link: $2^I \Rightarrow R$
- Quasi-convexity property
- Greedy-wise finding the shelled core
- Constraints defined by another monotone linkage function