Relational Similarity-Based Databases I

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Relational Similarity-Based Databases

general relational model of data:

- generalization of the classic RM (E. F. Codd)
- **similarity** relations on domains
- **ranks** assigned to tuples

motivation:

1. *similarity-based queries*
   "Show all houses that are sold for $600,000."
2. *approximate dependencies* in data
   "Do houses in similar locations have similar prices?"

goal:

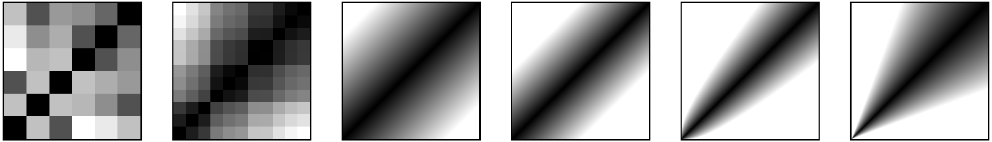
- rank-aware approach with solid logical foundations (logics of residuated structures)
- focus on all DB aspects (foundations, querying, dependencies, algorithms, ...)

Similarity-Based Query: An Example

id	*type*	*location*	*built*	*bdrm*	*sqft*	*price*
45	Single Family	Green St	1979	3	1180	$754,000
66	Ranch	Fulton St	1977	2	2400	$998,000
78	Single Family	Purdue Ave	1962	4	1360	$850,000
81	Residential	Hamilton Ave	1961	5	1450	$986,000
82	Condominium	Fulton St	1998	2	650	$540,000
87	Single Family	Bryant St	1927	3	1230	$854,000
95	Log Cabin	Schembri Ln	1936	2	750	$754,000
97	Penthouse	Cabrillo St	1984	1	932	$720,000

Similarity-Based Query: An Example

	id	*type*	*location*	*built*	*bdrm*	*sqft*	*price*
0.890	82	Condominium	Fulton St	1998	2	650	$540,000
0.595	97	Penthouse	Cabrillo St	1984	1	932	$720,000
0.535	87	Single Family	Bryant St	1927	3	1230	$854,000
0.487	66	Ranch	Fulton St	1977	2	2400	$998,000
0.472	45	Single Family	Green St	1979	3	1180	$754,000
0.277	81	Residential	Hamilton Ave	1961	5	1450	$986,000
0.213	95	Log Cabin	Schembri Ln	1936	2	750	$754,000



"Show houses located in Old Palo Alto and sold for $600,000."

Example (“Show houses with prices similar to $600,000” in SQL)

```
CREATE TABLE house (
  ⋮
  price NUMERIC NOT NULL
);

INSERT INTO house VALUES ···

CREATE FUNCTION sim (NUMERIC, NUMERIC) RETURNS NUMERIC AS
  'SELECT least (1, greatest (0, 1 + abs ($1 - $2) / -200000.0));'
  LANGUAGE SQL;

SELECT *, sim (price, 600000) AS rank
  FROM house
  ORDER BY sim DESC
  LIMIT 5;
```

Related work (1 of 2)

Fagin at al.

- R. Fagin. Combining fuzzy information: an overview.
  ACM SIGMOD Record 31(2):109–118, 2002.
- Natsev, Chang, Smith, Li, Vitter: Supporting incremental join queries on ranked inputs.
  In: *VLDB 2001*, pp. 281–290.
- Cohen, Sagiv: An incremental algorithm for computing ranked full disjunctions.
  In: *PODS 2005*, pp. 98–107.

RankSQL + related research

- Li, Chang, Ilyas, Song: RanSQL: Query Algebra and Optimization for Relational top-k queries.
  In: *ACM SIGMOD* 2005, pages 131–142, 2005.
- Illyas, Aref, Elmagarmid: Supporting top-k join queries in relational databases.
  The VLDB Journal 13:207–221, 2004.

Related work (2 of 2)

Extensions of Codd's model employing fuzzy logic

- several approaches (including "fuzzy data"), many papers
- Raju, Majumdar, Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems.
  ACM Trans. Database Systems 13:129–166, 1988.

Extensions of Codd's model employing probability

- **different** both semantically and technically (degrees of belief $\neq$ degrees of truth)
- D. Dey and S. Sarkar S. A probabilistic relational model and algebra.
  ACM Trans. Dat. Syst. 21:339–369, 1996.
- Fuhr, Rölleke, A probabilistic relational algebra for the integration of information retrieval and database systems.
  ACM Trans. Information Systems 15:32–66, 1997.
- Dalvi, Ré, Suciu, Probabilistic databases: diamonds in the dirt.
  Communications of the ACM 52:86–94, 2009.

Preliminaries from the Classic RM

attributes = *names for columns of ranked data tables*

- Y: denumerable set of all attributes
- attributes denoted $y, y', y_1, y_2, \ldots$

relation schemes = finite subsets $R \subseteq Y$

- relation schemes determine table columns (as in the Codd model)

cartesian (direct) product =

- set $\prod_{i \in I} A_i$ of all maps $f\colon I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$ (for given I-indexed set $\{A_i \mid i \in I\}$ of sets)

domains =

- sets of attribute values (D_y is domain of $y \in Y$)

tuples =

- elements of $\prod_{y \in R} D_y$ ($R \subseteq Y$)
- denoted $r \in \mathrm{Tupl}(R)$ (r is tuple on R over D_y's); $r(y)$ is called y-value of r

Motivation for Our Approach (1 of 3)

we want:

- *similarity-based queries* answered by *imprecise matches*

generalized RM:

- Shift from *two-element Boolean algebra* to (*complete*) *residuated lattices*
- Structure of matches in the classic RM $\Longrightarrow$ the generalized RM

starting with the classic RM: $\mathcal{D}$ on R can be viewed:

$$\mathcal{D} \colon \textstyle\prod_{y \in R} D_y \to \{0, 1\}$$

so that for only finitely many tuples $r \in \prod_{y \in R} D_y$: $\mathcal{D}(r) = 1$.

interpretation (if $\mathcal{D}$ is answer to Q)

$\mathcal{D}(r) = 1$ means "the tuple r matches the query Q"

$\mathcal{D}(r) = 0$ means "the tuple r does not match the query Q"

Motivation for Our Approach (2 of 3)

take a **partially ordered set** $\langle L, \leq, 0\rangle$ instead of $\langle\{0,1\}, \leq\rangle$:

$$\mathcal{D}\colon \prod_{y \in R} D_y \to L$$ (**ranked data table**, an RDT)

so that for only finitely many tuples $r \in \prod_{y \in R} D_y$: $\mathcal{D}(r) \neq 0$

desirable properties of L and $\leq$:

- lower and upper bound in L (0 for *no match*, 1 for *full match*),
- $\langle L, \leq\rangle$ is a complete lattice;
- additional operations on L to *aggregate degrees*.

conjunctive aggregator $\otimes$ motivated by *natural join* (for $\mathcal{D}_1$ on $R \cup S$ and $\mathcal{D}_2$ on $S \cup T$):

$$(\mathcal{D}_1 \bowtie \mathcal{D}_2)(rst) = \mathcal{D}_1(rs) \otimes \mathcal{D}_2(st),$$ (R, S, T are pairwise disjoint)

with $\otimes\colon \{0,1\}^2 \to \{0,1\}$ defined by $1 \otimes 1 = 1$ and $1 \otimes 0 = 0 \otimes 1 = 0 \otimes 0 = 0$

Motivation for Our Approach (3 of 3)

in our setting: $\otimes\colon L^2 \to L$ such that $\langle L, \otimes, 1\rangle$ is a commutative monoid and $\otimes$ is distributive w.r.t. $\bigvee$ (stronger condition than monotony):

$$a \otimes \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \otimes b_i)$$

which is equivalent to: $\langle L, \otimes, 1\rangle$ is a commutative monoid and there is (uniquely given) $\to\colon L^2 \to L$ such that

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \to c$$

(**adjointness property**)

altogether: $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a (**complete**) **residuated lattice**, i.e.

- $\langle L, \wedge, \vee, 0, 1\rangle$ … (complete) lattice,
- $\langle L, \otimes, 1\rangle$ … commutative monoid,
- $\langle \otimes, \to\rangle$ … adjoint pair ($a \otimes b \leq c$ iff $a \leq b \to c$).

Residuated Structures in Fuzzy Logics

- fuzzy logic in **broad sense**: any application of fuzzy approach in modeling
  - Zadeh L. A.: Fuzzy sets. *Inf. Control* (1965)
  - simple observations on handling of vagueness
- fuzzy logic in **narrow sense**: mathematical fuzzy logic
  - Hájek P.: *Metamathematics of Fuzzy Logic.* (1998)
  - Basic Logic (BL-logic), propositional/predicate; *logic of continuous t-norms*
  - Höhle, Esteva, Godo, Gottwald, Montagna, ...
  - various logical calculi (MTL-logic)

basic principles:

- adjointness derived from graded *modus ponens*
- propositions allowed to have "intermediate truth degrees", like:

$$||\text{value } x \text{ is similar to value } y||_{\mathbf{M}} = 0.9$$

- our case: $||\varphi||_{\mathbf{M},v}$ (φ formula; $\mathbf{M}$ database instance; v induced by tuples)

Domains with Similarities

similarity relations on domains (needed for approximate matches)
each domain D_y equipped with map $\approx_y : D_y \times D_y \to L$ satisfying:

(Ref) for each $d \in D_y$: $d \approx_y d = 1$,

(Sym) for each $d_1, d_2 \in D_y$: $d_1 \approx_y d_2 = d_2 \approx_y d_1$, and (optionally):

(Sep) for each $d_1, d_2 \in D_y$: $d_1 \approx_y d_2 = 1$ iff d_1 equals d_2, and

(Tra) for each $d_1, d_2, d_3 \in D_y$: $d_1 \approx_y d_2 \otimes d_2 \approx_y d_3 \leq d_1 \approx_y d_3$.

so-called **similarity relation**

domain with similarity $= \langle D_y, \approx_y \rangle$, where

- D_y is domain of attribute $y \in Y$;
- $\approx_y$ is similarity on D_y.

notes:

- interpretation: $u \approx_y v =$ degree to which u and v are similar
- boundary case: strict identity

Ranked Data Tables over Domains with Similarities

central notion to our model:

- formal counterpart to *relations on relation schemes* from Codd's model
- in mathematical fuzzy logic: interpretations of relation symbols

Definition (ranked data table)

Let $R \subseteq Y$ be a relation scheme and each $\langle D_y, \approx_y \rangle$ be a domain with similarity ($y \in R$). A **ranked data table on R over $\{\langle D_y, \approx_y \rangle \mid y \in R\}$** is any map $\mathcal{D}\colon \mathrm{Tupl}(R) \to L$ so that for only finitely many tuples $r \in \prod_{y \in R} D_y$: $\mathcal{D}(r) \neq 0$.

notes:

- RDTs are denoted $\mathcal{D}, \mathcal{D}', \mathcal{D}_1, \ldots$
- RDT on R over $\{\langle D_y, \approx_y \rangle \mid y \in R\}$ = fuzzy relation between D_y
- degree $\mathcal{D}(r)$ is called a **rank of r in $\mathcal{D}$**

Special Cases of RDTs

two important special cases:

Definition (RDTs on empty relation schemes)

For each $a \in L$, define $a_{\emptyset} = \{\langle \emptyset, a \rangle\}$.

Definition (singleton RDTs)

For each $y \in Y$ and $d \in D_y$, define $[y{:}d] = \{\langle \{\langle y, d \rangle\}, 1 \rangle\}$.

notes:

- $a_{\emptyset}$ is RDT on $R = \emptyset$ such that $a_{\emptyset}(\emptyset) = a$ (C. J. Date: $0_{\emptyset} = \texttt{TABLE\_DUM}$, $1_{\emptyset} = \texttt{TABLE\_DEE}$)
- $[y{:}d]$ is RDT on $R = \{y\}$ such that $[y{:}d](r) = \begin{cases} 1, & \text{if } r(y) = d, \\ 0, & \text{otherwise} \end{cases}$

Notes on Generalization of Codd's Model of Data

classic relational model results by:

- taking two-valued Boolean algebra for $\mathbf{L}$ (complete residuated lattice);
- considering each $\approx_y$ to be identity relation on D_y

consequence: all ranks become 1 (match) and 0 (no match)

nonranked RDT

- all ranks are from $\{0, 1\} \subseteq L$, $\mathbf{L}$ is arbitrary;
- stored data prior to querying;

Important feature of our model: stored data $=$ results of queries

RDTs represent both

- **stored data**, and
- **results of queries**.

Notes on Domain Similarities and Ranks

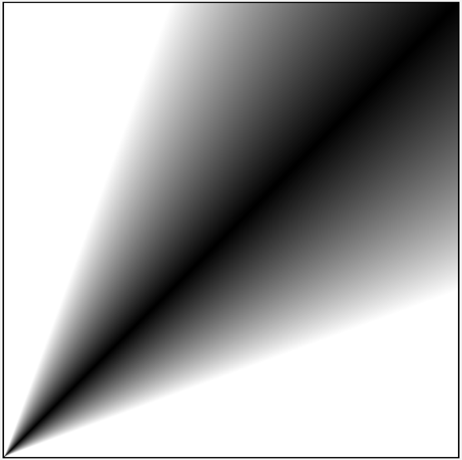
Where do similarities come from?

- can be assigned by an expert:
  - finite $\mathbf{L}$ or a finite subset of infinite linear $\mathbf{L}$;
  - Likert scale $L = \{1, \ldots, 5\}$ of degrees of satisfaction (Miller's 7 ± 2 phenomenon);
- can be determined based on "distance":
  - $\mathbf{L}$ on $[0, 1]$ with $\otimes$ being continuous Archimedean t-norm;
  - (pseudo)metric $\quad\Longmapsto\quad$ $\otimes$-transitive similarity;
- similarities are *purpose dependent*;
- implementation remark: can be stored (as data) / computed on demand.

Where do ranks come from?

- appear from nonranked data after performing similarity-based queries,
- can be assigned by experts,
- important aspect: *comparative meaning of truth degrees*.

Example (similarity on domain of "house prices")



$$d_1 \approx_{price} d_2 = s\big(|\log_b d_1 - \log_b d_2|\big)$$

$$b = 1 + 10^{-4}$$

$$s(x) = 1 - x \cdot 10^{-4}$$

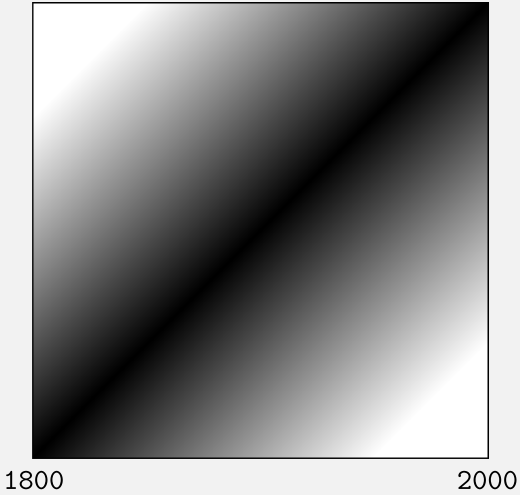
example:

$$\$1{,}000 \approx_{price} \$2{,}000 = 0.306$$

$$\$100{,}000 \approx_{price} \$101{,}000 = 0.990$$

$$\vdots \qquad \vdots$$

Example (similarity on domain of "construction years")



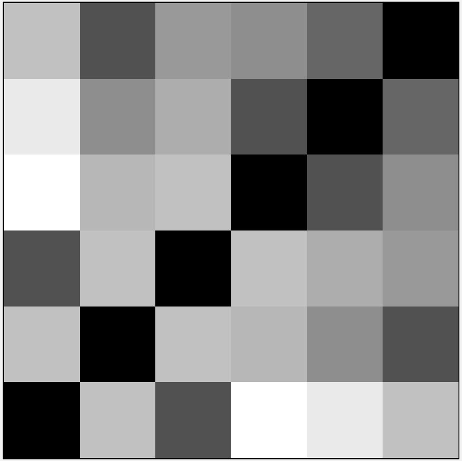
$$d_1 \approx_{year} d_2 = s\big(|d_1 - d_2|\big)$$

$$s(x) = 1 - x \cdot 150^{-1}$$

example:

$$1800 \approx_{year} 1840 = 0.733$$
$$1960 \approx_{year} 2000 = 0.733$$
$$\vdots \qquad \vdots$$

Example (similarity on domain of "property types")



Single Family

Residential

Ranch

Penthouse

Log Cabin

Condominium

Operations with RDTs

goal:

- propose set of (basic) operations with RDTs
- **purpose:** querying by performing operations with RTDs (relation algebra)
- **questions:** basic/derived operations, expressive power, …

groups of operations in our model:

- counterparts to boolean operations (union, intersection, residuum)
- natural join (and cross join)
- projection and residuated division
- similarity-based restrictions
- kernel and support
- renaming attributes

derived operations and extensions (II. part)

Counterparts to Boolean Intersection and Union

Definition

For RDTs $\mathcal{D}_1$ and $\mathcal{D}_2$ on relation scheme R, we define

$$(\mathcal{D}_1 \cup \mathcal{D}_2)(r) = \mathcal{D}_1(r) \vee \mathcal{D}_2(r),$$
$$(\mathcal{D}_1 \cap \mathcal{D}_2)(r) = \mathcal{D}_1(r) \wedge \mathcal{D}_2(r),$$
$$(\mathcal{D}_1 \otimes \mathcal{D}_2)(r) = \mathcal{D}_1(r) \otimes \mathcal{D}_2(r),$$

for all tuples r on R. $\mathcal{D}_1 \cup \mathcal{D}_2$ is called a **union** of $\mathcal{D}_1$ and $\mathcal{D}_2$; $\mathcal{D}_1 \cap \mathcal{D}_2$ and $\mathcal{D}_1 \otimes \mathcal{D}_2$ are called the **$\wedge$-intersection** and **$\otimes$-intersection** of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

idempotent vs. non-indempotent conjunction:

- RDT $\mathcal{D}$ on relation scheme R is called **idempotent (with respect to** $\otimes$**)** if $\mathcal{D} \otimes \mathcal{D} = \mathcal{D}$
- example: for $\mathcal{D}_1(r) = 0.5$ and $\mathcal{D}_2(r) = \cdots = \mathcal{D}_k(r) = 0.98$, we distinguish:
  - worst-match semantics: $(\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_k)(r) = 0.5$ (also if $\mathcal{D}_2(r) = \cdots = \mathcal{D}_k(r) = 0.5$)
  - all-match semantics: $(\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_k)(r) = 0.5 \cdot 0.98^{k-1}$ for Goguen $\otimes$
    $(\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_k)(r) = 0.5^k \lll 0.5 \cdot 0.98^{k-1}$ if $\mathcal{D}_2(r) = \cdots = \mathcal{D}_k(r) = 0.5$

Operations Based on Residuated Implication

issues with finiteness:

- componentwise application of $\rightarrow$: $(\mathcal{D}_1 \rightarrow \mathcal{D}_2)(r) = \mathcal{D}_1(r) \rightarrow \mathcal{D}_2(r)$
- if at least one D_y is infinite: $(\mathcal{D}_1 \rightarrow \mathcal{D}_2)(r) = 1$ for *infinitely many* r

(one possible) solution: for arbitrary degrees $a, b, c \in L$, define $b \twoheadrightarrow^{a} c \in L$ as follows:

$$b \twoheadrightarrow^{a} c = a \otimes (b \rightarrow c)$$

(**a-residuum of $b \in L$ with respect to $c \in L$**)

Definition (residuum of RDTs)

For RDTs $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ on R, we put

$$\big(\mathcal{D}_1 \twoheadrightarrow^{\mathcal{D}_3} \mathcal{D}_2\big)(r) = \mathcal{D}_1(r) \twoheadrightarrow^{\mathcal{D}_3(r)} \mathcal{D}_2(r)$$

for all tuples r. $\mathcal{D}_1 \twoheadrightarrow^{\mathcal{D}_3} \mathcal{D}_2$ is a **residuum** of $\mathcal{D}_1$ with respect to $\mathcal{D}_2$ which ranges over $\mathcal{D}_3$.

note:

- $\mathcal{D}_1 \twoheadrightarrow^{\mathcal{D}_3} \mathcal{D}_2 \subseteq \mathcal{D}_3$ (result of $\twoheadrightarrow$ in an RDT)

Theorem (properties of $\rightharpoondown$)

1. $b \rightharpoondown^1 c = b \rightarrow c$,
2. $1 \rightharpoondown^a c = 1 \rightharpoondown^c a = a \otimes c$,
3. $0 \rightharpoondown^a c = b \rightharpoondown^a 1 = a$,
4. $b \rightharpoondown^0 c = b \rightharpoondown^b 0 = 1 \rightharpoondown^b 0 = 0$,
5. $b \rightharpoondown^a c \leq b \rightharpoondown^1 (a \otimes c)$,
6. $\rightharpoondown$ *is monotone in the first and in the third argument,*
7. $\rightharpoondown$ *is antitone in the second argument,*
8. $a \rightharpoondown^a b \leq a \wedge b$,
9. *if* $\mathbf{L}$ *is divisible, then* $a \rightharpoondown^a b = a \wedge b$,
10. *if* $b \leq c$, *then* $b \rightharpoondown^a c = a$,
11. *if* $\mathbf{L}$ *is a linear* Π*-algebra, then* $b \leq c$ *iff* $b \rightharpoondown^a c = a$ *for all* $a > 0$,
12. $b \rightharpoondown^b c = c$ *iff there is* $x \in L$ *such that* $1 \rightharpoondown^x b = c$,
13. $1 \rightharpoondown^a b \leq c$ *iff* $a \leq b \rightharpoondown^1 c$.

$\langle \otimes, \rightarrow \rangle$ vs. $\rightarrowtriangle$

Theorem

Let $\mathbf{L} = \langle L, \wedge, \vee, \rightarrowtriangle, 0, 1 \rangle$ *be a structure such that* $\langle L, \wedge, \vee, 0, 1 \rangle$ *is a bounded lattice and* $\rightarrowtriangle$ *be a ternary operation satisfying the following conditions:*

$$
\begin{aligned}
1 \rightarrowtriangle^a 1 &= a, \\
1 \rightarrowtriangle^a b &= 1 \rightarrowtriangle^b a, \\
1 \rightarrowtriangle^a (1 \rightarrowtriangle^b c) &= 1 \rightarrowtriangle^c (1 \rightarrowtriangle^a b), \\
1 \rightarrowtriangle^a b \leq c \quad &\textit{iff} \quad a \leq b \rightarrowtriangle^1 c
\end{aligned}
$$

for all $a, b, c \in L$. *Then,* $\mathbf{L}' = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, *where* $a \otimes b = 1 \rightarrowtriangle^a b$ *and* $a \rightarrow b = a \rightarrowtriangle^1 b$ *for all* $a, b \in L$, *is a residuated lattice.*

corollary:

The class of all bounded lattices with $\rightarrowtriangle$ *satisfying the conditions above is a variety which is term equivalent to the variety of residuated lattices.*

Theorem (properties of operations $\otimes$, $\cap$, $\cup$, $\rightarrowtriangle$)

1. $\mathcal{D}_1 \otimes (\mathcal{D}_2 \cup \mathcal{D}_3) = (\mathcal{D}_1 \otimes \mathcal{D}_2) \cup (\mathcal{D}_1 \otimes \mathcal{D}_3)$

If $\mathbf{L}$ *is prelinear or divisible, then*

2. $\mathcal{D}_1 \otimes (\mathcal{D}_2 \cap \mathcal{D}_3) = (\mathcal{D}_1 \otimes \mathcal{D}_2) \cap (\mathcal{D}_1 \otimes \mathcal{D}_3)$,
3. $\mathcal{D}_1 \cap (\mathcal{D}_2 \cup \mathcal{D}_3) = (\mathcal{D}_1 \cap \mathcal{D}_2) \cup (\mathcal{D}_1 \cap \mathcal{D}_3)$.

If $\mathcal{D}$ *is nonranked, then*

4. $\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} (\mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3) = \mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$,
5. $(\mathcal{D}_1 \otimes \mathcal{D}_2) \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3 = \mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} (\mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$,
6. $\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2 = ((\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2) \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2) \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2$,
7. $\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} (\mathcal{D}_2 \cap \mathcal{D}_3) = (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2) \cap (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$,
8. $(\mathcal{D}_1 \cup \mathcal{D}_2) \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3 = (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3) \cap (\mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$,
9. $(\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2) \otimes (\mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3) \subseteq \mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3$.

If $\mathbf{L}$ *is prelinear, then*

10. $\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} (\mathcal{D}_2 \cup \mathcal{D}_3) = (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_2) \cup (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$,
11. $(\mathcal{D}_1 \cap \mathcal{D}_2) \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3 = (\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3) \cup (\mathcal{D}_2 \rightarrowtriangle^{\mathcal{D}} \mathcal{D}_3)$.

Natural Join

Definition (equality-based natural join)

If $\mathcal{D}_1$ is an RDT on relation scheme $R \cup S$ and $\mathcal{D}_2$ is an RDT of relation scheme $S \cup T$ such that $R \cap S = R \cap T = S \cap T = \emptyset$ (i.e., R, S, and T are pairwise disjoint), then the (**equality-based**) **natural join** of $\mathcal{D}_1$ and $\mathcal{D}_2$ is an RDT $\mathcal{D}_1 \bowtie \mathcal{D}_2$ on relation scheme $R \cup S \cup T$ defined by

$$\big(\mathcal{D}_1 \bowtie \mathcal{D}_2\big)(rst) = \mathcal{D}_1(rs) \otimes \mathcal{D}_2(st),$$

for each $r \in \mathrm{Tupl}(R)$, $s \in \mathrm{Tupl}(S)$, and $t \in \mathrm{Tupl}(T)$.

special cases:

- *cross join*: special case for $S = \emptyset$
- $\otimes$-*intersection*: special case for $R = \emptyset$ and $T = \emptyset$

basic properties:

- $\bowtie$ is *commutative* and *associative* (not idempotent in general); notation $\bowtie_{i=1}^{n} \mathcal{D}_i$
- $0_\emptyset$ is *annihilator*; $1_\emptyset$ is *neutral element*

Notes on Natural Joins

size of natural and cross joins:

- $|\mathcal{D}_1 \bowtie \mathcal{D}_2| \leq |\mathcal{D}_1| \cdot |\mathcal{D}_2|$
- but the converse inequality does not hold in general (not even in case of RDTs on disjoint relaiton schemes)

equality-based restriction via natural joins:

$$(\mathcal{D} \bowtie [y{:}d])(r) = \begin{cases} \mathcal{D}(r), & \text{if } r(y) = d, \\ 0, & \text{otherwise} \end{cases}$$

for all $r \in \mathrm{Tupl}(R)$

consequences:

- $\mathcal{D} \bowtie [y{:}d]$ = **equality-based restriction** of $\mathcal{D}$ consisting of tuples with y-values d
- ranks of those tuples in $\mathcal{D}$ are preserved

Projection

captures: existentially quantified queries (some A is B)

Definition (projection)

If $\mathcal{D}$ is an RDT on T, the **projection** $\pi_R(\mathcal{D})$ of $\mathcal{D}$ onto $R \subseteq T$ is defined by

$$(\pi_R(\mathcal{D}))(r) = \bigvee_{s \in \mathrm{Tupl}(T \setminus R)} \mathcal{D}(rs),$$

for each $r \in \mathrm{Tupl}(R)$.

special cases:

- $\big(\pi_\emptyset(\mathcal{D})\big)(\emptyset) = \bigvee_{t \in \mathrm{Tupl}(T)} \mathcal{D}(t)$
- $\pi_T(\mathcal{D}) = \mathcal{D}$ (if $\mathcal{D}$ is RDT on relation scheme T)

Theorem (selected properties of projection)

For any $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}$ *on* R*:*

1. *if* $R_1 \subseteq R_2$*, then* $\pi_{R_1}(\pi_{R_2}(\mathcal{D})) = \pi_{R_1}(\mathcal{D})$*,*
2. $\pi_R(\mathcal{D}_1 \cup \mathcal{D}_2) = \pi_R(\mathcal{D}_1) \cup \pi_R(\mathcal{D}_2)$*,*
3. $\pi_R(\mathcal{D}_1 \cap \mathcal{D}_2) \subseteq \pi_R(\mathcal{D}_1) \cap \pi_R(\mathcal{D}_2)$*,*
4. $\pi_R(\mathcal{D}_1 \otimes \mathcal{D}_2) \subseteq \pi_R(\mathcal{D}_1) \otimes \pi_R(\mathcal{D}_2)$*,*

Let $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be RDTs on relation schemes* $R \cup S$ *and* $S \cup T$ *such that* $R \cap S = R \cap T = S \cap T = \emptyset$*. Furthermore, let* $\{\mathcal{D}_i \,|\, i \in I\}$ *be a finite set of RDTs on* R_i $(i \in I)$*, and let* $\mathcal{D}$ *be an RDT on* $R = \bigcup_{i \in I} R_i$*. Then,*

5. $\pi_{R \cup S}(\mathcal{D}_1 \bowtie \mathcal{D}_2) = \mathcal{D}_1 \bowtie \pi_S(\mathcal{D}_2)$*,*
6. $\pi_{R_i}(\bowtie_{j \in I} \mathcal{D}_j) \subseteq \mathcal{D}_i$ *for all* $i \in I$*,*
7. $\mathcal{D}^{|I|} \subseteq \bowtie_{i \in I} \pi_{R_i}(\mathcal{D})$*,*
8. *if* $\mathcal{D}$ *is idempotent, then* $\mathcal{D} \subseteq \bowtie_{i \in I} \pi_{R_i}(\mathcal{D})$*.* □

semijoin: $\mathcal{D}_1 \ltimes \mathcal{D}_2 = \pi_{R \cup S}(\mathcal{D}_1 \bowtie \mathcal{D}_2) = \mathcal{D}_1 \bowtie \pi_S(\mathcal{D}_2)$ ($\otimes$ is distributive over $\bigvee$)

Residuated Division

captures: universaly quantified queries (all A's are B's)

Definition (residuated division)

Let $\mathcal{D}_1$ be an RDT on R, let $\mathcal{D}_2$ be an RDT on $S \subseteq R$, and let $\mathcal{D}_3$ be an RDT on $T = R \setminus S$. Then, a **division** $\mathcal{D}_1 \div^{\mathcal{D}_3} \mathcal{D}_2$ of $\mathcal{D}_1$ by $\mathcal{D}_2$ which ranges over $\mathcal{D}_3$ is an RDT on T defined by

$$\big(\mathcal{D}_1 \div^{\mathcal{D}_3} \mathcal{D}_2\big)(t) = \textstyle\bigwedge_{s \in \mathrm{Tupl}(S)} \big(\mathcal{D}_2(s) \rightarrowtriangle^{\mathcal{D}_3(t)} \mathcal{D}_1(st)\big),$$

for each $t \in \mathrm{Tupl}(T)$.

meaning:

$\mathcal{D}_2$ reliable suppliers, $\mathcal{D}_3$ solvent customers, $\mathcal{D}_1$ suppliers frequently used by customers, result = solvent customers frequently using all reliable suppliers

special cases:

- **graded containment**: $\big(\mathcal{D}_1 \div^{1_\emptyset} \mathcal{D}_2\big)(\emptyset) = \textstyle\bigwedge_{r \in \mathrm{Tupl}(R)} \big(\mathcal{D}_2(r) \rightarrow \mathcal{D}_1(r)\big)$

Derived Notions

subsethood and **similarity** degrees (note the role of $a_\emptyset$ and $a \in L$):

$$S(\mathcal{D}_1, \mathcal{D}_2) = \big(\mathcal{D}_2 \div^{1_\emptyset} \mathcal{D}_1\big)(\emptyset)$$
$$E(\mathcal{D}_1, \mathcal{D}_2) = S(\mathcal{D}_1, \mathcal{D}_2) \wedge S(\mathcal{D}_2, \mathcal{D}_1)$$

degrees of joinability:

Let $\mathcal{D}_i$ be RDTs on relation schemes R_i ($i \in I$ for finite I). Then

$$Jnd(\{\mathcal{D}_i \,|\, i \in I\}) = \textstyle\bigwedge_{i \in I} S\big(\mathcal{D}_i, \pi_{R_i}(\bowtie_{j \in I} \mathcal{D}_j)\big)$$

is a **degree of joinability** of RDTs $\mathcal{D}_i$ ($i \in I$);
RDTs $\mathcal{D}_i$ ($i \in I$) **join completely** if $Jnd(\{\mathcal{D}_i \,|\, i \in I\}) = 1$

degrees of decomposability:

Let $\mathcal{D}$ be an RDT on relation schemes $R = \bigcup_{i \in I} R_i$ where I is finite. Then

$$Dcd(\mathcal{D}, \{R_i \,|\, i \in I\}) = E\big(\mathcal{D}, \bowtie_{i \in I} \pi_{R_i}(\mathcal{D})\big)$$

is a **degree of decomposability** of $\mathcal{D}$ with respect to R_i ($i \in I$);
$\mathcal{D}$ has a **nonloss decomposition** if $Dcd(\mathcal{D}, \{R_i \,|\, i \in I\}) = 1$

Concept-Forming Operators Induced by RDTs

Definition

For an RDT $\mathcal{D}_1$ on R; $S \subseteq R$, $T = R \setminus S$; and nonranked RDTs $\mathcal{D}_y$ on $\{y\}$ ($y \in R$), put

$$f^{S,T}_{\mathcal{D}_1,\{\mathcal{D}_y \,|\, y \in R\}}(\mathcal{D}_2) = \mathcal{D}_1 \div^{\bowtie_{y \in T} \mathcal{D}_y} \mathcal{D}_2$$

for any $\mathcal{D}_2$ on S.

notes:

- $\mathcal{D}_1$ and $\mathcal{D}_y$ ($y \in R$) induce $f^{S,T}_{\mathcal{D}_1,\{\mathcal{D}_y \,|\, y \in R\}}$ with respect to S and T (in this order)
- **dyadic case**: for $R = \{x, y\}$, $\mathcal{D}_x$, $\mathcal{D}_y$, $\mathcal{D} \subseteq \mathcal{D}_x \bowtie \mathcal{D}_y$, $\mathcal{D}_A \subseteq \mathcal{D}_x$, and $\mathcal{D}_B \subseteq \mathcal{D}_y$:

$$f^{\{x\},\{y\}}_{\mathcal{D},\{\mathcal{D}_x,\mathcal{D}_y\}}(\mathcal{D}_A) = \mathcal{D} \div^{\mathcal{D}_y} \mathcal{D}_A, \qquad f^{\{y\},\{x\}}_{\mathcal{D},\{\mathcal{D}_x,\mathcal{D}_y\}}(\mathcal{D}_B) = \mathcal{D} \div^{\mathcal{D}_x} \mathcal{D}_B,$$

  express concept-forming operators (denoted by $^{\uparrow}$ and $^{\downarrow}$) used in the dyadic FCA of object-attribute relational data with graded attributes (generalizes to n-adic case)

Similarity-Based Restriction

Definition (similarity-based restriction)

For any attributes $y_1, y_2 \in R$ with the same domains with similarity we define the **similarity-based restriction** $\sigma_{y_1 \approx y_2}(\mathcal{D})$ of $\mathcal{D}$ by $y_1 \approx y_2$ which is an RDT on R defined by

$$(\sigma_{y_1 \approx y_2}(\mathcal{D}))(r) = \mathcal{D}(r) \otimes r(y_1) \approx_{y_1} r(y_2),$$

for all $r \in \mathrm{Tupl}(R)$.

representation by natural joins: $\sigma_{y_1 \approx y_2}(\mathcal{D}) = \mathcal{D} \bowtie \mathcal{D}_{y_1 \approx y_2}$, where for all $r \in \mathrm{Tupl}(R)$,

$$\mathcal{D}_{y_1 \approx y_2}(r(\{y_1, y_2\})) = \begin{cases} r(y_1) \approx_{y_1} r(y_2), & \text{if } \mathcal{D}(r) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

restriction based on domain values:

$$(\sigma_{y \approx d}(\mathcal{D}))(r) = \mathcal{D}(r) \otimes r(y) \approx_y d$$

derived operation:

$$\sigma_{y \approx d}(\mathcal{D}) = \pi_R(\sigma_{y \approx y'}(\mathcal{D} \bowtie [y'{:}d])).$$

Theorem (properties of similarity-based restrictions)

The following are true (if both left and right-hand sides exist):

1. $\pi_S(\sigma_{y\approx z}(\mathcal{D})) = \sigma_{y\approx z}(\pi_S(\mathcal{D}))$ *if* $\mathcal{D}$ *is an RDT on* R *and* $R \cap \{y, z\} \subseteq S$,
2. $\sigma_{y\approx z}(\mathcal{D}_1 \bowtie \mathcal{D}_2) = \sigma_{y\approx z}(\mathcal{D}_1) \bowtie \mathcal{D}_2$ *if* $\mathcal{D}_2$ *is an RDT on* R_2 *and* $\{y, z\} \cap R_2 = \emptyset$,
3. $\sigma_\theta(\mathcal{D}_1 \cup \mathcal{D}_2) = \sigma_\theta(\mathcal{D}_1) \cup \sigma_\theta(\mathcal{D}_2)$,
4. $\sigma_\theta(\mathcal{D}_1 \cap \mathcal{D}_2) \subseteq \sigma_\theta(\mathcal{D}_1) \cap \mathcal{D}_2$,
5. $\sigma_\theta(\mathcal{D}_1 \otimes \mathcal{D}_2) = \sigma_\theta(\mathcal{D}_1) \otimes \mathcal{D}_2$,
6. $\mathcal{D}_1 \rightarrowtriangle^{\sigma_\theta(\mathcal{D}_3)} \mathcal{D}_2 = \sigma_\theta(\mathcal{D}_1 \rightarrowtriangle^{\mathcal{D}_3} \mathcal{D}_2)$.

If $\mathbf{L}$ *is prelinear or divisible, then*

7. $\sigma_\theta(\mathcal{D}_1 \cap \mathcal{D}_2) = \sigma_\theta(\mathcal{D}_1) \cap \sigma_\theta(\mathcal{D}_2)$,
8. $\mathcal{D}_1 \div^{\sigma_\theta(\mathcal{D}_3)} \mathcal{D}_2 = \sigma_\theta(\mathcal{D}_1 \div^{\mathcal{D}_3} \mathcal{D}_2)$.

Kernel and Support

Definition (kernel and support)

For any RDT $\mathcal{D}$ on relation scheme R, the **kernel** $\Delta\mathcal{D}$ and **support** $\nabla\mathcal{D}$ of $\mathcal{D}$ are RDTs on R defined by

$$(\Delta\mathcal{D})(r) = \begin{cases} 1, & \text{if } \mathcal{D}(r) = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad (\nabla\mathcal{D})(r) = \begin{cases} 1, & \text{if } \mathcal{D}(r) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $r \in \mathrm{Tupl}(R)$.

notes:

- express non-ranked RDT from general ones
- notation by M. Baaz (projections and relativizations)
- *kernel* (interior operator); $\Delta\mathcal{D}$ is the greatest nonranked RDT such that $\Delta\mathcal{D} \subseteq \mathcal{D}$
- *support* (closure operator); $\nabla\mathcal{D}$ is the least nonranked RDT such that $\mathcal{D} \subseteq \nabla\mathcal{D}$
- two borderline cases of other possibilities (monotone and indepotent operators)

Theorem (properties of Δ and ∇)

The following are true (if both left and right-hand sides exist):

1. $\Delta\mathcal{D}_1 \otimes \mathcal{D}_2 = \Delta\mathcal{D}_1 \otimes \Delta\mathcal{D}_2$, $\nabla\mathcal{D}_1 \otimes \mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \otimes \nabla\mathcal{D}_2$,
2. $\Delta\mathcal{D}_1 \cap \mathcal{D}_2 = \Delta\mathcal{D}_1 \cap \Delta\mathcal{D}_2$, $\nabla\mathcal{D}_1 \cap \mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \cap \nabla\mathcal{D}_2$,
3. $\Delta\mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \Delta\mathcal{D}_1 \cup \Delta\mathcal{D}_2$, $\nabla\mathcal{D}_1 \cup \mathcal{D}_2 = \nabla\mathcal{D}_1 \cup \nabla\mathcal{D}_2$,
4. $\Delta\mathcal{D}_1 \rightharpoonup^{\mathcal{D}_3} \mathcal{D}_2 \subseteq \Delta\mathcal{D}_1 \rightharpoonup^{\Delta\mathcal{D}_3} \Delta\mathcal{D}_2$,
   $\Delta\mathcal{D}_1 \rightharpoonup^{\mathcal{D}_3} \mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \rightharpoonup^{\Delta\mathcal{D}_3} \nabla\mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \rightharpoonup^{\nabla\mathcal{D}_3} \nabla\mathcal{D}_2$
5. $\Delta\mathcal{D}_1 \bowtie \mathcal{D}_2 = \Delta\mathcal{D}_1 \bowtie \Delta\mathcal{D}_2$, $\nabla\mathcal{D}_1 \bowtie \mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \bowtie \nabla\mathcal{D}_2$,
6. $\Delta\pi_R(\mathcal{D}) \supseteq \pi_R(\Delta\mathcal{D})$, $\nabla\pi_R(\mathcal{D}) = \pi_R(\nabla\mathcal{D})$,
7. $\Delta\mathcal{D}_1 \div^{\mathcal{D}_3} \mathcal{D}_2 \subseteq \Delta\mathcal{D}_1 \div^{\Delta\mathcal{D}_3} \Delta\mathcal{D}_2$,
   $\Delta\mathcal{D}_1 \div^{\mathcal{D}_3} \mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \div^{\Delta\mathcal{D}_3} \nabla\mathcal{D}_2 \subseteq \nabla\mathcal{D}_1 \div^{\nabla\mathcal{D}_3} \nabla\mathcal{D}_2$
8. $\Delta\sigma_\theta(\mathcal{D}) \subseteq \sigma_\theta(\Delta\mathcal{D})$, $\Delta\sigma_\theta(\mathcal{D}) = \Delta\sigma_\theta(\Delta\mathcal{D})$.

If $\mathbf{L}$ *is linear, then*

9. $\nabla\mathcal{D}_1 \cap \mathcal{D}_2 = \nabla\mathcal{D}_1 \cap \nabla\mathcal{D}_2$, $\Delta\mathcal{D}_1 \cup \mathcal{D}_2 = \Delta\mathcal{D}_1 \cup \Delta\mathcal{D}_2$,
10. $\Delta\pi_R(\mathcal{D}) = \pi_R(\Delta\mathcal{D})$.

□

Renaming

usual operation of renaming attributes:

Definition (renaming attributes)

For an RDT $\mathcal{D}$ on R and an injective map $h\colon R \to Y$ such that for all $y \in R$, the attributes $h(y)$ and y have identical domains with equalities, we define a **renaming** $\rho_h(\mathcal{D})$ of $\mathcal{D}$ by h as an RDT on $h(R) = \{h(y) \mid y \in R\}$ by $(\rho_h(\mathcal{D}))(h(r)) = \mathcal{D}(r)$, where $h(r) \in \mathrm{Tupl}(h(R))$ such that $(h(r))(h(y)) = r(y)$ for each attribute $y \in R$.

notation: $\rho_{h(y_1),\ldots,h(y_n)\leftarrow y_1,\ldots,y_n}(\mathcal{D})$ means $\rho_h(\mathcal{D})$ if $R = \{y_1, \ldots, y_n\}$

- we omit ith component in $y_1, \ldots, y_n \leftarrow h(y_1), \ldots, h(y_n)$ whenever $h(y_i) = y_i$

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To Be Continued . . .

second part:

- types, domains, database instances
- formalization of queries
- relation algebra as query language
- domain relational calculus
- relational completeness
- derived operations
- further extensions
- notes