Attribute Dependencies for Data with Grades II

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Problem Setting

input: data table $\langle X, Y, I \rangle$

output: collection T of graded attribute implications such that:

1. T is reasonably small (simple, ...), and
2. T conveys information about exactly all graded implications true in data

questions:

- what is a *collection?* (a set / $\mathbf{L}$-set of formulas)
- what is *reasonably small?* (in terms of size, redundancy, ...)
- what is meant by *true in data?* (fully true, true to a degree, ...)

in case you missed it: goto DAMOL webpage

Belohlavek, Vychodil: *Attribute Dependencies for Data with Grades I*

Completeness in $\langle X, Y, I \rangle$

Definition (completeness in data)

A set T of graded attribute implications is called **complete in $\langle \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{I} \rangle$** if

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I \rangle}$$

for every implication $A \Rightarrow B$.

verbally: T being complete in $\langle X, Y, I \rangle$ means:

> *"each graded attribute implication $A \Rightarrow B$ semantically follows from T to the degree to which it is true in $\langle X, Y, I \rangle$"*

notes:

- T is an (ordinary) set
- completeness characterizes *all degrees* (strong property)

Theorem

T is complete in $\langle X, Y, I\rangle$ iff $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$.

Proof (sketch).

Let T be complete. If $M \in \mathrm{Mod}(T)$, then for $M \Rightarrow M^{\downarrow\uparrow}$,

$$||M \Rightarrow M^{\downarrow\uparrow}||_M \geq ||M \Rightarrow M^{\downarrow\uparrow}||_T = ||M \Rightarrow M^{\downarrow\uparrow}||_{\langle X,Y,I\rangle} = S(M^{\downarrow\uparrow}, M^{\downarrow\uparrow}) = 1,$$

i.e., $M \in \mathrm{Int}(X^*, Y, I)$ because $M^{\downarrow\uparrow} \subseteq M$.

Conversely, for $M \in \mathrm{Int}(X^*, Y, I)$ and $A \Rightarrow B \in T$:

$$||A \Rightarrow B||_M \geq ||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_T = 1,$$

i.e., $M \in \mathrm{Mod}(T)$.

The "$\Leftarrow$"-part is easy to see:

$$\begin{aligned} ||A \Rightarrow B||_T &= \bigwedge\{||A \Rightarrow B||_M \mid M \in \mathrm{Mod}(T)\} \\ &= \bigwedge\{||A \Rightarrow B||_M \mid M \in \mathrm{Int}(X^*, Y, I)\} = ||A \Rightarrow B||_{\langle X,Y,I\rangle}. \end{aligned}$$

□

Example (Complete Sets)

Let $\langle X, Y, I\rangle$ be any data table with graded attributes.

Then $T = \{A \Rightarrow A^{\downarrow\uparrow} \mid A \in L^Y\}$ is complete in $\langle X, Y, I\rangle$.

Apply the previous Theorem:

$\mathrm{Mod}(T) \subseteq \mathrm{Int}(X^*, Y, I)$-part:

for $M \Rightarrow M^{\downarrow\uparrow} \in T$, we get $1 = S(M, M)^* \leq S(M^{\downarrow\uparrow}, M)$, i.e., $M^{\downarrow\uparrow} \subseteq M$.

$\mathrm{Mod}(T) \supseteq \mathrm{Int}(X^*, Y, I)$-part:

if $M = M^{\downarrow\uparrow}$, then $S(A, M)^* \leq S(A^{\downarrow\uparrow}, M^{\downarrow\uparrow}) = S(A^{\downarrow\uparrow}, M)$, i.e., $||A \Rightarrow A^{\downarrow\uparrow}||_M = 1$.

Furthermore, $T = \{A \Rightarrow B \mid ||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1\}$ is complete in $\langle X, Y, I\rangle$

both "$\subseteq$" and "$\supseteq$" are easy to see

1-Completeness in $\langle X, Y, I \rangle$

Definition (1-completeness in data)

A set T of graded attribute implications is called a **1-complete in $\langle X, Y, I \rangle$** if

$$||A \Rightarrow B||_T = 1 \text{ iff } ||A \Rightarrow B||_{\langle X,Y,I \rangle} = 1$$

for every implication $A \Rightarrow B$.

interesting characterization of completeness:

"it suffices to characterize all $A \Rightarrow B$ true to degree 1":

Theorem

T is complete in $\langle X, Y, I \rangle$ iff T is 1-complete in $\langle X, Y, I \rangle$.

Proof (sketch).

"$\Rightarrow$" is trivial, "$\Leftarrow$": is proven by showing that $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$. □

Redundant Theories

question:

"What is a reasonably small T?"

Definition (redundant sets of graded implications)

A set T of graded implications is **redundant** if there exists $A \Rightarrow B \in T$ such that $||A \Rightarrow B||_{T-\{A \Rightarrow B\}} = 1$. Otherwise, call T is **non-redundant**.

Lemma

The following conditions are equivalent:

1. *T is a non-redundant set of implications;*
2. *For every $A \Rightarrow B \in T$: $\mathrm{Mod}(T) \subset \mathrm{Mod}(T - \{A \Rightarrow B\})$;*
3. *For every $A \Rightarrow B \in T$ there are C, D s. t. $||C \Rightarrow D||_{T-\{A \Rightarrow B\}} < ||C \Rightarrow D||_T$.* □

Non-redundant Bases of $\langle X, Y, I \rangle$

Definition (non-redundant bases)

A set T of graded implications is called a **non-redundant base of $\langle X, Y, I \rangle$** if T is complete in $\langle X, Y, I \rangle$ and no proper subset of T is complete in $\langle X, Y, I \rangle$.

based on characterization of redundant sets:

Corollary

T is a non-redundant base of $\langle X, Y, I \rangle$ if and only if

- *T is complete in $\langle X, Y, I \rangle$, and*
- *T is non-redundant as a set of implications.* □

Systems of Pseudo-Intents

naive approach to getting non-redundant bases:

- start with a complete set,
- remove all redundant implications (based on checking entailment degrees)

alternatively describe (certain) non-redundant bases directly:

Definition (system of pseudo-intents)

$\mathcal{P} \subseteq L^Y$ is a **system of pseudo-intents of $\langle X, Y, I\rangle$** if for each $P \in L^Y$:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P.$$

equivalent formulations: $\mathcal{P} \subseteq L^Y - \mathrm{Int}(X^*, Y, I)$ such that

$$P \in \mathcal{P} \quad \text{iff} \quad P \in \mathrm{Mod}(\{Q \Rightarrow Q^{\downarrow\uparrow} \,|\, Q \in \mathcal{P} \text{ and } Q \neq P\})$$

$$\text{iff} \quad P \in \mathrm{Mod}(T - \{P \Rightarrow P^{\downarrow\uparrow}\}), \text{ where } T = \{Q \Rightarrow Q^{\downarrow\uparrow} \,|\, Q \in \mathcal{P}\}$$

Example (Systems of Pseudo-Intents)

multiple systems of pseudo-intents:

Structure of degrees and data:

- $L = \{0, 0.5, 1\}$, linearly ordered, Gödel $\otimes$ (i.e., $\otimes = \wedge$); * is identity;
- $X = \{x\}$, $Y = \{y, z\}$, $I(x, y) = I(x, z) = 0$

Systems of pseudo-intents:

$$\mathcal{P}_1 = \{\{z\}, \{{}^{0.5}/y, {}^{0.5}/z\}, \{y\}\}, \qquad \mathcal{P}_3 = \{\{y\}, \{{}^{0.5}/z\}\},$$
$$\mathcal{P}_2 = \{\{z\}, \{{}^{0.5}/y\}\}, \qquad \mathcal{P}_4 = \{\{{}^{0.5}/y\}, \{{}^{0.5}/z\}\}.$$

no system of pseudo-intents:

Structure of degrees:

- L on $[0, 1]$ given by left-continuous t-norm; * is globalization;
- $X = \{x\}$, $Y = \{y\}$, $I(x, y) = 0$

Theorem

If $\mathcal{P}$ is a system of pseudo-intents of $\langle X, Y, I\rangle$ then

$$T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$$

is a non-redundant base of $\langle X, Y, I\rangle$.

Proof (sketch).

T is complete: each intent is obviously a model of T (previous example). Conversely, if $M \in \mathrm{Mod}(T)$, then $||P \Rightarrow P^{\downarrow\uparrow}||_M = 1$ for all $P \in \mathcal{P}$. Now, if $M \in \mathcal{P}$, we get $M \notin \mathrm{Mod}(T)$ since $||M \Rightarrow M^{\downarrow\uparrow}||_M < 1$. Therefore, $M \notin \mathcal{P}$, i.e., $M = M^{\downarrow\uparrow}$ by definition of $\mathcal{P}$.

T is non-redundant: take $M \in \mathcal{P}$; directly from $||P \Rightarrow P^{\downarrow\uparrow}||_M = 1$ for all $P \in \mathcal{P}$ such that $P \neq M$, we get $\mathrm{Mod}(T) \subset \mathrm{Mod}(T - \{M \Rightarrow M^{\downarrow\uparrow}\})$. □

Graphs Induced by $\langle X, Y, I\rangle$

Definition (graph induced by data)

For $\langle X, Y, I\rangle$, define

$$V = \{P \in L^Y \mid P \neq P^{\downarrow\uparrow}\}.$$

If $V \neq \emptyset$, define

$$E = \{\langle P, Q\rangle \in V \times V \mid P \neq Q \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P \neq 1\}$$

and the corresponding **graph** $\mathbf{G} = \langle \boldsymbol{V}, \boldsymbol{E} \cup \boldsymbol{E}^{-1}\rangle$.

For any $Q \in V$ and $\mathcal{P} \subseteq V$ define the following subsets of V:

$$\mathrm{Pred}(Q) = \{P \in V \mid \langle P, Q\rangle \in E\},$$
$$\mathrm{Pred}(\mathcal{P}) = \bigcup\{\mathrm{Pred}(Q) \mid Q \in \mathcal{P}\}.$$

trivial case: if $V = \emptyset$ then $\mathcal{P} = \emptyset$ (and $T = \emptyset$ is the non-redundant base)

Graph-Theoretic Interpretation of Pseudo-Intents

Theorem

Let $\mathcal{P} \subseteq V$. $\mathcal{P}$ is a system of pseudo-intents iff $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$.

Lemma

Let $\emptyset \neq \mathcal{P} \subseteq V$. If $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$ then $\mathcal{P}$ is a maximal independent set in $\mathbf{G}$.

Corollary

The following are equivalent:

1. *$\mathcal{P} \neq \emptyset$ is a system of pseudo-intents;*
2. *$\mathcal{P}$ is a maximal independent set in $\mathbf{G}$ such that $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$.* □

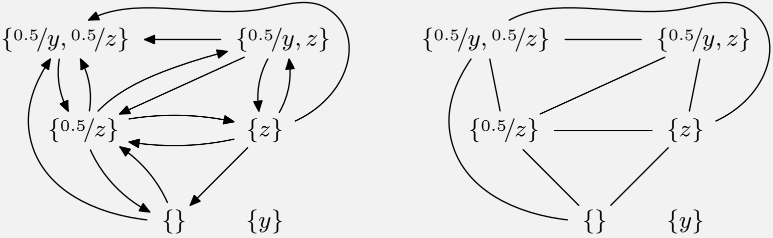
getting all systems of pseudo-intents:

- list all maximal independent sets $\mathcal{P}$ in $\mathbf{G}$ satisfying $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$

Example (Graph-Theoretic Interpretation of Pseudo-Intents)

Structure of degrees and data:

- $L = \{0, 0.5, 1\}$, linearly ordered, Łukasiewicz $\otimes$; * is identity;
- $X = \{x\}$, $Y = \{y, z\}$, $I(x, y) = 0.5$, $I(x, z) = 1$.



Maximal independent sets and **systems of pseudo-intents**:

$\mathcal{P}_1 = \{\{\}, \{{}^{0.5}/y, z\}, \{y\}\}$,

$\mathcal{P}_2 = \{\{{}^{0.5}/z\}, \{y\}\}$,

$\mathcal{P}_3 = \{\{z\}, \{y\}\}$,

$\mathcal{P}_4 = \{\{{}^{0.5}/y, {}^{0.5}/z\}, \{y\}\}$.

Notes on Pseudo-Intents and Globalization

systems of pseudo-intents:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P$$

in case of globalization:

$$\begin{aligned} ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \quad & \text{iff} \quad \text{either } Q \nsubseteq P \text{ or } Q \subseteq P \text{ and } Q^{\downarrow\uparrow} \subseteq P \\ & \text{iff} \quad Q \subseteq P \text{ implies } Q^{\downarrow\uparrow} \subseteq P \end{aligned}$$

definition of $\mathcal{P}$ simplifies to:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for all } Q \subset P \text{ such that } Q \in \mathcal{P}$$

case of finite $\boldsymbol{Y}$ and $\boldsymbol{L}$:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for all } Q \subset P \text{ such that } Q \in \mathcal{P}$$

consequence: $\mathcal{P}$ exists and is unique

Minimal Bases (Case of Globalization)

Definition (minimal base of $\langle X, Y, I\rangle$)

A set T of graded implications is called a **minimal base of $\langle \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{I}\rangle$** if T is complete in $\langle X, Y, I\rangle$ and $|T| \leq |T'|$ for each set T' which is complete in $\langle X, Y, I\rangle$.

case of globalization:

Theorem

Let $\mathbf{L}$ be a finite residuated lattice with globalization.
If Y is finite, then $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ is minimal.

Proof (idea).

Using the fact that for each $P \in \mathcal{P}$, there is $A \Rightarrow B \in T'$ such that $A^{\downarrow\uparrow} = P^{\downarrow\uparrow}$, considering $f : \mathcal{P} \to T'$ such that $f(P) = A \Rightarrow B$ and proving it is injective. □

Algorithm (Graph-Based Method for Globalization)

Data: $E \subseteq V \times V$, list $\mathcal{S}$ of lectically ordered element from V
Result: system of pseudo-intents $\mathcal{P}$ (for L and Y finite)

set $\mathcal{P}$ **to** $\emptyset$
while $\mathcal{S} \neq \emptyset$ **do**
    $\mathcal{P} := \mathcal{P} \cup \mathrm{first}(\mathcal{S})$
    vacate $\mathcal{S}'$
    for B **in** $\mathrm{rest}(\mathcal{S})$ **do**
        if $\langle B, \mathrm{first}(\mathcal{S}) \rangle \notin E$ **then**
            append B **to** $\mathcal{S}'$
        end
    end
    $\mathcal{S} := \mathcal{S}'$
end
return $\mathcal{P}$

Further Properties (for Globalization)

closure operator: define cl_T by iteration of f (Tarski, Cousot & Cousot):

$$f(M) = M \cup \bigcup\{B \otimes S(A, M)^* \mid A \Rightarrow B \in T \text{ and } A \neq M\}$$
$$= M \cup \bigcup\{B \mid A \Rightarrow B \in T \text{ and } A \subset M\}$$

Theorem

For finite Y and L with globalization, $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$, the set of all fixed points of cl_T if $\mathcal{P} \cup \mathrm{Int}(X^*, Y, I)$. □

computational issues:

- cl_T uses (subsets of) T to compute (furter formulas of) T
- fixpoints shall be enumerated in an order extending "$\subset$"

Algorithm (Enumerating Fixed Points of cl_T for Globalization)

Data: data table $\langle X, Y, I\rangle$
Result: system of pseudo-intents $\mathcal{P}$ (for L and Y finite)

$B := \emptyset$
$T := \emptyset$
while $B \neq Y$ **do**
  if $B \neq B^{\downarrow\uparrow}$ **then**
    $T := T \cup \{B \Rightarrow B^{\downarrow\uparrow}\}$
  end
  $B := \mathrm{NextClosure}(B, cl_T)$
end
return T

optional extension:

computes $\mathcal{P}$ *and* $\mathrm{Int}(X^*, Y, I)$ *(additional else-branch)*

Getting Non-redundant Bases by Reduction

alternative approach: for general * (Y and L finite), introduce $\mathcal{P}$ by putting

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for all } Q \subset P \text{ such that } Q \in \mathcal{P}$$

consequence:

Theorem

Under the above notation, $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ *is complete.* □

two-step procedure:

- compute T (as in case of globalization but with general *)
- remove redundant implications from T

Summary

results on graded attribute implications:

- foundations,
- semantic entailment,
- complete ordinary-style and graded-style axiomatizations,
- reductions to ordinary attribute imlications,
- relationship to other types of dependencies,
- non-redundant bases,
- algorithms
- more results available, ...