# Ordinally equivalent data: a measurement-theoretic look at formal concept analysis of fuzzy attributes

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# Ordinally equivalent data: a measurement-theoretic look at formal concept analysis of fuzzy attributes

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**Abstract**

We show that if two fuzzy relations, representing data tables with graded attributes, are ordinally equivalent then their concept lattices with respect to the Gödel operations on chains are (almost) isomorphic and that the assumption of Gödel operations is essential. We argue that measurement-theoretic results like this one are important for pragmatic reasons in relational data modeling and outline issues for future research.

*Key words:* concept lattice, fuzzy logic, degree, measurement

## 1. Introduction and problem setting

A frequent objection to using degrees in representing vague terms such as "tall" can be articulated as follows. Why to assign the truth degree 0.764 to the proposition "John is tall"? Why not 0.682? This objection has a clear pragrmatic aspect and suggests a fundamental problem in using truth degrees. The objection is found in various forms in the literature on vagueness, see e.g. [29, pp. 52–53] and also [11, 16, 17], and in many debates since the inception of fuzzy logic.

Whether and to what extent this objection, appealing as it is, indeed presents a problem, calls for close scrutiny. Presumably, one needs to look for answers pertaining to the usage of truth degrees in general as well as those that apply to particular models and applications. In our view, the issues involved are naturally looked at from the viewpoint of the theory of measurement. Measurement theory has been initiated by [31], in which the so-called ordinal, interval and ratio scales were recognized, and further developed in many publications within mathematical psychology, see e.g. [10, 19, 23, 25, 28].

In this paper, we examine some of the questions offered by the above considerations in a limited scope of a particular area, namely formal concept analysis (FCA). Limited as it is, formal concept analysis encompasses rather general structures such as lattices, closure structures and operators, and Galois connections, hence the ramifications are broad. The basic problem we consider may be described as follows. Consider the following table, representing fuzzy relation $I_1$ between objects $x_1$, $x_2$ and $x_3$, and attributes $y_1, \ldots, y_4$.

| $I_1$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|---|---|---|---|
| $x_1$ | 1 | 0.9 | 0.8 | 1 |
| $x_2$ | 0 | 1 | 0.5 | 0.5 |
| $x_3$ | 0.8 | 0.8 | 0.2 | 0.1 |

To what extent do the values of truth degrees, i.e. 1, 0.9, 0.8, etc. matter? What happens if we replace 0.8 by 0.7 in the three entries in the table? This question is important from a pragmatic viewpoint. Namely, when filling in the table, by a domain expert or a data analyst, one needs to know about the impact of the values and their relationships on further processing of the table. Since the basic structures utilized in FCA are concept lattices derived from such data tables, we are particularly interested in the impact on the structure of the concept lattice corresponding to the given data table.

In our previous work [1], later extended to the framework of general relational structures of first-order fuzzy logic [5], we showed then with an appropriately defined notion of similarity, the following claim can be proven: the degree of similarity of two data tables is less than or equal to the degree of similarity ot the corresponding concept lattices, i.e. similar data tables lead to similar concept lattices. Hence, in a sense, the exact values of the truth degrees do not actually matter as far as the associated concept lattice is concerned.

In this paper, we examine a related but different issue. It consists in considering as essential the ordering of truth degrees, rather than the particular (numerical) values representing them. This view is implicitly present in describing fuzzy logic as a "logic of comparative truth". To make our point more concrete, consider as a simple example three propositions, $\varphi_1$, $\varphi_2$, and $\varphi_3$, and two truth valuations, $e_1$ and $e_2$, corresponding to two experts. Let $e_1(\varphi_1) = 0.2$, $e_1(\varphi_2) = 0.5$, $e_1(\varphi_3) = 0.9$, and $e_2(\varphi_1) = 0.15$, $e_2(\varphi_2) = 0.63$, $e_2(\varphi_3) = 0.8$. Even though the degrees assigned to the same proposition by the two experts are different, and one sometimes has $e_1(\varphi_i) < e_2(\varphi_i)$ and sometimes $e_1(\varphi_i) > e_2(\varphi_i)$, there is still an important kind of consistency of $e_1$ with $e_2$. Namely, for every pair $\varphi_i$ and $\varphi_j$ of propositions we have

$$e_1(\varphi_i) \leq e_1(\varphi_j) \text{ if and only if } e_2(\varphi_i) \leq e_2(\varphi_j).$$

Similar kind of consistency in using degrees of membership was reported in experimental work on the psychology of concepts in the early 1970s [22, 26, 27]. Continuing with our example, one might call the expert assignments $e_1$ and $e_2$ ordinally equivalent and ask whether and under which conditions a further processing based on $\varphi_1$, $\varphi_2$, and $\varphi_3$ corresponding to the two truth valuations results in two consistent conclusions.

In this paper, we define the notion of ordinal equivalence for data tables with fuzzy attributes and prove that when using the Gödel logic connectives on linearly ordered sets of degrees, the concept lattices associated to ordinally equivalent data tables are almost isomorphic (see Remark 1) with the corresponding formal concepts pairwise ordinally equivalent. In addition, if the tables are even strongly ordinally equivalent, the concept lattices are isomorphic. We describe the isomorphisms and prove that the assumption of Gödel operations is essential. Results of this kind are important in addressing the issues regarding the significance of the values of truth degrees and the choice of fuzzy logic connectives in formal concept analysis as well as in a broader context of fuzzy logic models. The preliminary notions are surveyed in Section 2. Section 3 presents the results. We conclude the paper by a summary and a brief outline of future research issues.

## 2. Preliminaries

*Structures of truth degrees.* As a scale of truth degrees we use a complete residuated lattice [13, 14, 15], i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of $L$, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy the adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$. Elements $a$ of $L$ are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm [14] with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). Namely, all other continuous t-norms are obtained as ordinal sums of these three [14, 15]. Alternatively, we can take a finite subset $L \subseteq [0, 1]$ equipped with appropriate operations. Having $\mathbf{L}$ as the structure of truth degrees, we use the usual notions of fuzzy sets and fuzzy relations [2, 14, 32].

*Formal concept analysis of data with fuzzy attributes.* Let $X$ and $Y$ be finite non-empty sets of objects and attributes, respectively, $I$ be a fuzzy relation between $X$ and $Y$. That is, $I : X \times Y \rightarrow L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ to which the object $x$ has the attribute $y$. The triplet $\langle X, Y, I\rangle$, called a *formal* $\mathbf{L}$*-context*, represents a data table, such as the one shown above, with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x, y)$.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X}(A(x) \rightarrow I(x, y)) \text{ and } B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y}(B(y) \rightarrow I(x, y)).$$

Using basic rules of predicate fuzzy logic, $A^{\uparrow}(y)$ is the truth degree of "for each $x \in X$: if $x$ belongs from $A$ then $x$ has $y$". Similarly for $B^{\downarrow}$. That is, $A^{\uparrow}$ is a fuzzy set of attributes common to all objects of $A$, and $B^{\downarrow}$ is a fuzzy set of objects sharing all attributes of $B$. The set

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$

denoted also just by $\mathcal{B}(I)$, of all fixpoints of $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ thus contains all pairs $\langle A, B \rangle$ such that $A$ is the collection of all objects that have all the attributes of $B$, and $B$ is the collection of all attributes that are shared by all the objects of $A$. Elements $\langle A, B \rangle \in \mathcal{B}(X, Y, I)$ will be called *formal concepts* of $\langle X, Y, I \rangle$; $A$ and $B$ are called the extent and intent of $\langle A, B \rangle$, respectively; $\mathcal{B}(X, Y, I)$ is called the **L**-*concept lattice* of $\langle X, Y, I \rangle$. Both the extent $A$ and the intent $B$ are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not necessarily 0 and 1.

For $\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle \in \mathcal{B}(X, Y, I)$, put

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1\text{)}.$$

This defines a *subconcept-superconcept* hierarchy on $\mathcal{B}(X, Y, I)$. The structure of $\mathcal{B}(X, Y, I)$ is described by the so-called main theorem for fuzzy concept lattices. We only mention that $\mathcal{B}(X, Y, I)$ equipped with $\leq$ is a complete lattice where infima and suprema are given by

$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow} \rangle \,, \tag{1}$$

$$\bigvee_{j \in J} \langle A_j, B_j \rangle = \langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j \rangle \,. \tag{2}$$

For more information we refer to e.g. [4, 8, 20, 24, 30].

## 3. Ordinally equivalent data tables and their concept lattices

The kind of consistency alluded to above may be formalized as follows. We say that fuzzy relations $I_1$ and $I_2$ between $X$ and $Y$ are *ordinally equivalent*, in symbols $I_1 \equiv I_2$, if for every $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have

$$I_1(x_1, y_1) \leq I_1(x_2, y_2) \text{ iff } I_2(x_1, y_1) \leq I_2(x_2, y_2).$$

We also need the following, stronger variant of $\equiv$. $I_1$ and $I_2$ are *strongly ordinally equivalent*, in symbols $I_1 \equiv_{\{1\}} I_2$, if

$$I_1 \equiv I_2 \text{ and for every } x_1, x_2 \in X, y_1, y_2 \in Y\colon\ I_1(x, y) = 1 \text{ iff } I_2(x, y) = 1.$$

Clearly, $\equiv$ may be defined for fuzzy sets in general, by putting for $A, B \in L^U$, $A \equiv B$ iff $A(u) \leq A(v)$ iff $B(u) \leq B(v)$ for every $u, v \in U$. From a different point of view, let for $A$ define a binary relation $\leq_A$ in $U$ by $u \leq_A v$ iff $A(u) \leq A(v)$. Then $\leq_A$ is a quasiorder and $A \equiv B$ is equivalent to the fact that $\leq_A$ coincides with $\leq_B$.

If $I_1$ and $I_2$ represent two expert opinions, $I_1 \equiv I_2$ means that agree on whether object $x_1$ has attribute $y_1$ to a higher degree than to which object $x_2$ has attribute $y_2$, for all objects and attributes. $I_1 \equiv_{\{1\}} I_2$ means that, in addition, the experts agree on when attributes fully apply to objects.

**Example 1.** Consider the following data tables.

| $I$ | $y_1$ | $y_2$ |
|---|---|---|
| $x_1$ | $3/4$ | $1/4$ |
| $x_2$ | $0$ | $1/4$ |

| $J$ | $y_1$ | $y_2$ |
|---|---|---|
| $x_1$ | $1$ | $1/4$ |
| $x_2$ | $0$ | $1/4$ |

| $K$ | $y_1$ | $y_2$ |
|---|---|---|
| $x_1$ | $1$ | $1/2$ |
| $x_2$ | $0$ | $1/2$ |

| $M$ | $y_1$ | $y_2$ |
|---|---|---|
| $x_1$ | $1$ | $1/4$ |
| $x_2$ | $0$ | $0$ |

$I$ and $J$ are ordinally equivalent, i.e. $I \equiv J$, but not strongly ordinally equivalent, i.e. $I \not\equiv_{\{1\}} J$ because $I(x_1, y_1) = {}^3\!/_4$ while $J(x_1, y_1) = 1$. $J$ and $K$ are even strongly ordinally equivalent, i.e. $J \equiv_{\{1\}} K$. None of $I$, $J$, and $K$ is ordinally equivalent with $M$ because while $M(x_2, y_2) \leq M(x_2, y_1)$, we have $I(x_2, y_2) \not\leq I(x_2, y_1)$ and the same for $J$ and $K$.

The following example is instructive for our examination.

**Example 2.** Let $L = \{0, {}^1\!/_3, {}^2\!/_3, 1\}$. The following fuzzy relations clearly satisfy $I_1 \equiv_{\{1\}} I_2$.

| $I_1$ | $y_1$ | $y_2$ |
|---|---|---|
| $x$ | $0$ | ${}^1\!/_3$ |

| $I_2$ | $y_1$ | $y_2$ |
|---|---|---|
| $x$ | ${}^1\!/_3$ | ${}^2\!/_3$ |

While the concept lattices $\mathcal{B}_{\mathrm{G}}(I_1)$ and $\mathcal{B}_{\mathrm{G}}(I_2)$ of $I_1$ and $I_2$ are isomorphic when we equip $L$ with the Gödel operations, the concept lattices $\mathcal{B}_{\mathrm{L}}(I_1)$ and $\mathcal{B}_{\mathrm{L}}(I_2)$ with respect to the Łukasiewicz operations are not. This follows from the fact that formal concepts are uniquely determined by their intents and that the four concept lattices involved have the following intents ($Y$ denotes$\{\{{}^1/y_1\}, \{{}^1/y_2\}\}$):

$$\begin{aligned}
&\mathcal{B}_{\mathrm{G}}(I_1) : \{\{{}^0/y_1\}, \{{}^{\frac{1}{3}}/y_2\}\}, \{\{{}^0/y_1\}, \{{}^1/y_2\}\}, \text{ and } Y,\\
&\mathcal{B}_{\mathrm{G}}(I_2) : \{\{{}^{\frac{1}{3}}/y_1\}, \{{}^{\frac{2}{3}}/y_2\}\}, \{\{{}^{\frac{1}{3}}/y_1\}, \{{}^1/y_2\}\}, \text{ and } Y,\\
&\mathcal{B}_{\mathrm{L}}(I_1) : \{\{{}^0/y_1\}, \{{}^{\frac{1}{3}}/y_2\}\}, \{\{{}^{\frac{1}{3}}/y_1\}, \{{}^{\frac{2}{3}}/y_2\}\}, \{\{{}^{\frac{2}{3}}/y_1\}, \{{}^1/y_2\}\}, \text{ and } Y,\\
&\mathcal{B}_{\mathrm{L}}(I_2) : \{\{{}^{\frac{1}{3}}/y_1\}, \{{}^1/y_2\}\}, \{\{{}^{\frac{2}{3}}/y_1\}, \{{}^1/y_2\}\}, \text{ and } Y.
\end{aligned}$$

That is, for the Łukasiewicz operations, the concept lattices have different numbers of formal concepts.

As we show next, this example is no coincidence. In particular, we show that ordinally equivalent $I_1$ and $I_2$ lead to isomorphic concept lattices if $I_1 \equiv_{\{1\}} I_2$ or almost isomorphic (in a sense made precise in Theorem 3 and Remark 1) concept lattices for $I_1 \equiv I_2$ when $L$ is equipped with the Gödel operations on linearly ordered sets of degrees. Looking at the results the other way around, they imply that one should use the Gödel operations if one requires that ordinally equivalent data imply isomorphic concept lattices.

Unless otherwise stated, we assume from now on that the complete residuated lattice $\mathbf{L}$ is linearly ordered and is equipped with Gödel operations. That is, $a \leq b$ or $b \leq a$ for every $a, b \in L$ and

$$\begin{aligned}
a \otimes b &= a \wedge b,\\
a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b,\\ b & \text{if } a > b. \end{cases}
\end{aligned}$$

In what follows, we utilize the fact that ordinal equivalence of $I_1$ and $I_2$ means that either of $I_1$ and $I_2$ may be brought to the other one by means of an increasing bijection of the degrees involved (note that this claim holds for linearly as well as non-linearly ordered $L$). More precisely, let for $i = 1, 2$,

$$I_i(X, Y) = \{I_i(x, y) \mid x \in X, y \in Y\}. \tag{3}$$

**Lemma 1.** *$I_1 \equiv I_2$ if and only if there exists an increasing bijection $f : I_1(X,Y) \to I_2(X,Y)$ such that*

$$I_2 = f \circ I_1,$$

*i.e. $I_2(x,y) = f(I_1(x,y))$ for every $x$ and $y$. For $I_1 \equiv_{\{1\}} I_2$, the corresponding condition for $f$ is stronger in that $f(1) = 1$ whenever $1 \in I_1(X,Y)$ or $1 \in I_2(X,Y)$.*

*Proof.* If $I_1 \equiv I_2$ then the required $f$ is defined by $f(I_1(x,y)) = I_2(x,y)$, for every $x \in X$ and $y \in Y$. This definition is correct because the ordinal equivalence of $I_1$ and $I_2$ and the antisymmetry of the ordering of truth degrees imply that $I_1(x,y) = I_1(x',y')$ is equivalent to $I_2(x,y) = I_2(x',y')$. The converse claim is obvious. □

Because for $I_1 \equiv I_2$ the function $f$ from Lemma 1 is uniquely determined, we call it the function corresponding to $I_1$ and $I_2$ and denote it also by $f_{I_1,I_2}$ in what follows. Furthermore, for a function $f : I_1(X,Y) \to I_2(X,Y)$, we consider the function

$$f^+ : I_1(X,Y) \cup \{1\} \to I_2(X,Y) \cup \{1\}$$

defined by

$$f^+(a) = \begin{cases} f(a) & \text{if } a \in \mathrm{dom}(f), \\ 1 & \text{if } a = 1 \text{ and } 1 \notin \mathrm{dom}(f). \end{cases}$$

where $\mathrm{dom}(f)$ denotes the set of degrees for which $f$ is defined. For a mapping $h : L_1 \to L_2$ and a fuzzy set $A \in L_1^U$, we define a fuzzy set $h(A) \in L_2^U$ by

$$(h(A))(u) = h(A(u)).$$

We first consider the stronger assumption of $I_1 \equiv_{\{1\}} I_2$.

**Theorem 1.** *If $I_1 \equiv_{\{1\}} I_2$ then the mapping $g$ defined by*

$$g(A,B) = \langle f^+_{I_1,I_2}(A), f^+_{I_1,I_2}(B)\rangle$$

*is an isomorphism of $\mathcal{B}(X,Y,I_1)$ to $\mathcal{B}(X,Y,I_2)$. Moreover, if $g(A,B) = \langle C,D\rangle$ then $A \equiv_{\{1\}} C$ and $B \equiv_{\{1\}} D$.*

*Proof.* Put $L_1 = I_1(X,Y) \cup \{1\}$ and $L_2 = I_2(X,Y) \cup \{1\}$.

First, observe that for any $A \in L^X$ we have $A^{\uparrow_{I_1}}(y) \in L_1$ for each $y \in Y$. Indeed, for any $x \in X$ we have either $A(x) \leq I_1(x,y)$ or $A(x) > I_1(x,y)$. In the former case, $A(x) \to I_1(x,y) = 1 \in L_1$, in the latter case, $A(x) \to I_1(x,y) = I_1(x,y) \in L_1$. Due to finiteness of $X$ we have

$$A^{\uparrow_{I_1}}(y) = \textstyle\bigwedge_{x\in X} A(x) \to I_1(x,y) = \min_{x\in X} A(x) \to I_1(x,y) \in L_1.$$

Similarly we obtain $A^{\uparrow_{I_2}}(y) \in L_2$, and $B^{\downarrow_{I_1}}(y) \in L_1$ and $B^{\downarrow_{I_2}}(y) \in L_2$ for every $B \in L^Y$ and each $x \in X$.

It is easily observed that both $L_1$ and $L_2$ are closed under the operations of the original $\mathbf{L}$. Therefore, $L_1$ and $L_2$, equipped with the restrictions of the operations of $\mathbf{L}$ form complete residuated lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ (with the provision that if 0 does not belong to $L_i$, then $0_i$ is the least element of $L_i$ for $i = 1,2$).

The assumption $I_1 \equiv_{\{1\}} I_2$ moreover implies that $f^+_{I_1,I_2}$ is a (complete) lattice isomorphism of $L_1$ into $L_2$, because $f^+_{I_1,I_2}$ is clearly a bijection and, moreover, $a \le b$ for $a, b \in L_1$ means $a = I_1(x_1, y_1) \le I_1(x_2, y_2) = b$ for some $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, which is equivalent to $f^+_{I_1,I_2}(a) = I_2(x_1, y_1) \le I_2(x_2, y_2) = f^+_{I_1,I_2}(b)$ due to $I_1 \equiv_{\{1\}} I_2$. Moreover, $f^+_{I_1,I_2}$ preserves $\to$. Indeed, either $a \le b$ and then $f^+_{I_1,I_2}(a) \le f^+_{I_1,I_2}(b)$ from which we get $f^+_{I_1,I_2}(a \to b) = f^+_{I_1,I_2}(1) = 1 = f^+_{I_1,I_2}(a) \to f^+_{I_1,I_2}(b)$, or $a > b$ and $f^+_{I_1,I_2}(a) \le f^+_{I_1,I_2}(b)$ and then $f^+_{I_1,I_2}(a \to b) = f^+_{I_1,I_2}(b) = f^+_{I_1,I_2}(a) \to f^+_{I_1,I_2}(b)$.

Now, Theorem 3.2 of [3] implies that $g$ is an onto lattice homomorphism of the $\mathbf{L}_1$-concept lattice $\mathcal{B}(X, Y, I_1)$ onto the $\mathbf{L}_2$-concept lattice $\mathcal{B}(X, Y, I_2)$. But since $L_1$ and $L_2$ are subsets of $L$ closed under the operations of $\mathbf{L}$, $\mathcal{B}(X, Y, I_1)$ and $\mathcal{B}(X, Y, I_2)$ are also $\mathbf{L}$-concept lattices. Furthermore, since $f^+_{I_1,I_2}$ is a bijection, $g$ is clearly a bijection, too, and hence an isomorphism of the $\mathbf{L}$-concept lattices $\mathcal{B}(X, Y, I_1)$ and $\mathcal{B}(X, Y, I_2)$.

The facts $A \equiv_{\{1\}} C$ and $B \equiv_{\{1\}} D$ are immediate. The proof is complete. □

**Example 3.** Consider the fuzzy relations $J$ and $K$ from Example 1. Recall that $J \equiv_{\{1\}} K$. The bijection $f_{J,K} : J(X, Y) \to K(X, Y)$, i.e. the mapping $f$ from Lemma 1, is given by

$$f_{J,K}(0) = 0,\ f_{J,K}({}^1\!/\!_4) = {}^1\!/\!_2, \quad \text{and} \quad f_{J,K}(1) = 1.$$

Clearly, $f^+_{J,K}$ coincides with $f_{J,K}$. One may verify that

$$\mathcal{B}(X, Y, J) = \{\langle \tfrac{1}{4}0, 11\rangle, \langle 10, 1\tfrac{1}{4}\rangle, \langle \tfrac{1}{4}\tfrac{1}{4}, 01\rangle, \langle 11, 0\tfrac{1}{4}\rangle\},$$

where $\langle \frac{1}{4}0, 11\rangle$ stands for the formal concept $\langle A, B\rangle$ for which $A(x_1) = {}^1\!/\!_4$, $A(x_2) = 0$, $B(y_1) = 1$, and $B(y_2) = 1$; similarly for the other concepts. According to Theorem 1, $f^+_{J,K}$ provides an isomorphism of $\mathcal{B}(X, Y, J)$ to $\mathcal{B}(X, Y, K)$. Hence, $\mathcal{B}(X, Y, K)$ consists of the formal concepts

$$\langle \tfrac{1}{2}0, 11\rangle = f^+_{J,K}(\langle \tfrac{1}{4}0, 11\rangle), \langle 10, 1\tfrac{1}{2}\rangle = f^+_{J,K}(\langle 10, 1\tfrac{1}{4}\rangle),$$
$$\langle \tfrac{1}{2}\tfrac{1}{2}, 01\rangle = f^+_{J,K}(\langle \tfrac{1}{4}\tfrac{1}{4}, 01\rangle), \langle 11, 0\tfrac{1}{2}\rangle\} = f^+_{J,K}(\langle 11, 0\tfrac{1}{4}\rangle).$$

The following theorem shows that, as far as finite case is considered, no other than the Gödel operations have the property from Theorem 1.

**Theorem 2.** *Let* $\mathbf{L}$ *be a finite linearly ordered residuated lattice. If* $\otimes$ *is different from* min*, then there exist fuzzy relations* $I_1, I_2 \in L^{X\times Y}$ *such that* $I_1 \equiv_{\{1\}} I_2$ *and* $\mathcal{B}(X, Y, I_1)$ *and* $\mathcal{B}(X, Y, I_2)$ *are not isomorphic.*

*Proof.* Let us first prove that there exist $q, r \in L$ such that

$$q > r \text{ and } q \to r > r. \tag{4}$$

Assume the contrary, i.e. that for every $q > r$ we have $q \to r = r$ (this is indeed the contrary because we always have $q \to r \ge r$). Let us recall [2] that in every complete residuated lattice, $p \otimes q = \bigwedge\{r \mid p \le q \to r\}$. Without loss of generality, assume $p \leq q$. Then

$$\begin{aligned} p \otimes q &= \bigwedge \{r \mid p \leq q \to r\} = \\ &= \bigwedge \{r \mid q > r,\, p \leq q \to r\} \wedge \bigwedge \{r \mid q \leq r,\, p \leq q \to r\} = \\ &= \bigwedge \{r \mid q > r,\, p \leq r\} \wedge q = \\ &= \begin{cases} 1 \wedge q = q = \min(p,q) & \text{if } p = q, \\ p \wedge q = \min(p,q) & \text{if } p < q, \end{cases} \end{aligned}$$

contradicting the assumption that $\otimes$ is different from min.

Let now $b$ be the largest $r$ for which a $q$ exists satisfying (4), and let $a$ be the largest $q$ for this $b$ for which (4) holds, i.e. $a \to b > b$. Condider the tables

$$\begin{array}{c|c} I_1 & y \\ \hline x & b \end{array} \quad \text{and} \quad \begin{array}{c|c} I_2 & y \\ \hline x & a \end{array}$$

Since $1 \to r = r$, we have $a \neq 1$, and hence also $b \neq 1$. Therefore, $I_1 \equiv_{\{1\}} I_2$. We show that $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_2)$ have different numbers of elements and are thus not isomorphic. In particular, we show that $\mathcal{B}(X,Y,I_1)$ contains at least three formal concepts while $\mathcal{B}(X,Y,I_2)$ only two.

Indeed, recall that every formal concept $\langle A, B\rangle$ is uniquely determined by its intent $B$ and that the intents are just all fuzzy sets in $Y$ of the form $C^{\uparrow}$ for some $C \in L^X$. For $\alpha \in [0,b]$, we have $\alpha \to I_1(x,y) = \alpha \to b = 1$, hence $\{{}^{\alpha}/x\}^{\uparrow_{I_1}} = \{{}^{1}/y\}$. For $\alpha = a$, we have $a \to I_1(x,y) = a \to b$, hence $\{{}^{\alpha}/x\}^{\uparrow_{I_1}} = \{{}^{a\to b}/y\}$. For $\alpha \in (a,1]$, which is nonempty due to $a \neq 1$, we have $\alpha \to b = b$, since first, $\alpha \to b \geq b$ is always the case, and second, $\alpha > a$ and $a$ is the largest one for which $a \to b > b$. Hence, $\{{}^{\alpha}/x\}^{\uparrow_{I_1}} = \{{}^{b}/y\}$. Therefore, as 1, $a \to b$, and $b$ are mutually different, $\mathcal{B}(X,Y,I_1)$ contains at least three formal concepts. Note that the fact that $1 \neq b$ is established above, $a \to b \neq b$ follows from the assumption (4) regarding $a$ and $b$, particularly from $a \to b > b$, and $a \to b \neq 1$ follows again from the assumption (4) regarding $a$ and $b$, particularly from $a > b$, because $a \to b = 1$ would imply $a \leq b$ due to adjointness.

Now, for $\alpha \in [0,a]$, we have $\alpha \to I_2(x,y) = \alpha \to a = 1$, hence $\{{}^{\alpha}/x\}^{\uparrow_{I_2}} = \{{}^{1}/y\}$. For $\alpha \in (a,1]$, we have $\alpha \to I_2(x,y) = \alpha \to a = a$, because we always have $\alpha \to a \geq a$ and because $\alpha \to a > a$ does not hold. Namely, we have $a > b$ and by assumption, $b$ is the largest one for which there exists $q$ exists such that $q \to b > b$. Hence, $\{{}^{\alpha}/x\}^{\uparrow_{I_2}} = \{{}^{a}/y\}$. As a result, $\mathcal{B}(X,Y,I_2)$ contains exactly two formal concepts. □

Next, we consider the weaker assumption of $I_1 \equiv I_2$ instead of $I_1 \equiv_{\{1\}} I_2$. Let thus $I_1 \equiv I_2$ but not $I_1 \equiv_{\{1\}} I_2$. Then there exist $x \in X$, $y \in Y$, and $a \in L$ such that either $I_2(x,y) = 1$ $I_1(x,y) = a < 1$, or $I_1(x,y) = 1$ and $I_2(x,y) = a < 1$. We assume the former, i.e. assume that $x$, $y$, and $a$ satisfy

$$a = I_1(x,y). \tag{5}$$

Clearly, $I_1 \equiv I_2$ implies that for every $x'$ and $y'$, $I_1(x',y') = a$ if and only if $I_2(x',y') = 1$.

Let us denote by $I_1^+$ the fuzzy relation resulting from $I_1$ by replacing all occurences of $a$ by 1, i.e.

$$I_1^+(x,y) = \begin{cases} 1 & \text{if } I_1(x,y) = a, \\ I_1(x,y) & \text{if } I_1(x,y) \neq a. \end{cases} \tag{6}$$

Furthermore, let for $\langle A, B\rangle \in \mathcal{B}(X,Y,I_1^+)$ denote by $\langle A_-, B_-\rangle$ and $\langle A^-, B^-\rangle$ the pairs of fuzzy sets defined by

$$A_-(x) = \begin{cases} a & \text{if } A(x) = 1, \\ A(x) & \text{if } A(x) \neq 1; \end{cases} \qquad B_- = B; \tag{7}$$

and

$$A^- = A; \qquad B^-(x) = \begin{cases} a & \text{if } B(x) = 1, \\ B(x) & \text{if } B(x) \neq 1. \end{cases} \tag{8}$$

Recall that the 1-cut ${}^1C$ of a fuzzy set $C$ in universe $U$ is the ordinary set ${}^1C$ defined by

$${}^1C = \{u \in U \mid C(u) = 1\}.$$

We need the following assertions.

**Lemma 2.** *Let $\langle A, B\rangle \in \mathcal{B}(X,Y,I_1^+)$.*

(1) *If ${}^1A \neq \emptyset$ then $\langle A^-, B^-\rangle \in \mathcal{B}(X,Y,I_1)$.*

(2) *If ${}^1B \neq \emptyset$ then $\langle A_-, B_-\rangle \in \mathcal{B}(X,Y,I_1)$.*

(3) *${}^1A \neq \emptyset$ or ${}^1B \neq \emptyset$.*

*Proof.* (1) and (2) are symmetric, hence we prove only (2). We need to verify $A_-^{\uparrow_{I_1}} = B_-$ and $B_-^{\downarrow_{I_1}} = A_-$.

First, we show $A_-^{\uparrow_{I_1}} = B_-$. We have

$$\begin{aligned} A_-^{\uparrow_{I_1}}(y) &= \bigwedge_{x \in X} (A_-(x) \to I_1(x,y)) = \\ &= \bigwedge_{x \notin {}^1A} (A_-(x) \to I_1(x,y)) \wedge \bigwedge_{x \in {}^1A} (A_-(x) \to I_1(x,y)). \end{aligned} \tag{9}$$

For $x \notin {}^1A$ we have $A_-(x) = A(x) < a$. Indeed, since $A(x) = B^{\downarrow_{I_1^+}}(x)$ and since $Y$ is finite, $A(x) = B(y') \to I_1^+(x,y')$ for some $y'$, hence $A(x) = I_1^+(x,y')$ due to $A(x) \neq 1$ and the properties of $\to$. Now $I_1^+(x,y') = A(x) < 1$ because $I_1^+(x,y') \neq 1$ implies $I_1^+(x,y') = I_1(x,y')$ and for such $\langle x, y'\rangle$ the ordinal equivalence of $I_1$ and $I_2$ implies $I_1(x,y') < a$, hence also $I_1^+(x,y') < a$. Since $A(x) = I_1^+(x,y')$, we conclude $A(x) < a$. The latter fact also implies $A_-(x) = A(x)$. We now get

$$A_-(x) \to I_1(x,y) = A(x) \to I_1^+(x,y), \tag{10}$$

for if $I_1^+(x,y) \neq 1$ then $I_1(x,y) = I_1^+(x,y)$; while if $I_1^+(x,y) = 1$ then $I_1(x,y) = a$, whence we have $A_-(x) \leq I_1(x,y)$ and $A(x) \leq I_1^+(x,y)$ from which we get $A_-(x) \to I_1(x,y) = A(x) \to I_1^+(x,y) = 1$.

For $x \in {}^1A$ we have $A_-(x) = a$. Furthermore, if $I_1^+(x,y) \neq 1$ we get $I_1(x,y) = I_1^+(x,y) < a$ as above and so

$$A_-(x) \to I_1(x,y) = a \to I_1(x,y) = I_1(x,y) = I_1^+(x,y) = A(x) \to I_1^+(x,y); \quad (11)$$

if $I_1^+(x,y) = 1$ we have $I_1(x,y) = a$ and so

$$A_-(x) \to I_1(x,y) = a \to I_1(x,y) = 1 = A(x) \to I_1^+(x,y). \quad (12)$$

Now, (9) along with (10), (11), (12), and the assumption $\langle A, B\rangle \in \mathcal{B}(X, Y, I_1^+)$ imply

$$\begin{aligned} A_-^{\uparrow_{I_1}}(y) &= \bigwedge_{x \notin {}^1A} (A(x) \to {I_1}^+(x,y)) \wedge \bigwedge_{x \in {}^1A} (A(x) \to {I_1}^+(x,y)) = \\ &= A_-^{\uparrow_{I_1^+}}(y) = B(y) = B_-(y), \end{aligned}$$

proving $A^{\uparrow_{I_1}} = B_-$.

Next we show $B_-^{\uparrow_{I_1}} = A_-$. We distinguish two cases, $x \notin {}^1A$ and $x \in {}^1A$. First, let $x \notin {}^1A$: Since $A(x) < 1$, we obtain similarly as above that $A(x) < a$, i.e. $A_-(x) = A(x)$. Therefore, the finiteness of $Y$ implies that there exists $y \in Y$ such that

$$B(y) \to I_1^+(x,y) = B^{\downarrow_{I_1^+}}(x) = A(x).$$

$A(x) < a$ implies $B(y) > I_1^+(x,y) = A(x)$, hence $I_1^+(x,y) < a$. By definition, $I_1^+(x,y) < a$ implies $I_1(x,y) = I_1^+(x,y)$. Since $B(y) > I_1^+(x,y)$, we have $B(y) > I_1(x,y)$, hence

$$B(y) \to I_1(x,y) = I_1(x,y) = A(x). \quad (13)$$

Now observe that

$$B_-^{\downarrow_{I_1}}(x) = B(y) \to I_1(x,y). \quad (14)$$

Indeed, for this equality, "$\leq$" is obvious and "$<$" leads to a contradiction. Namely, $B_-^{\downarrow_{I_1}}(x) < B(y) \to I_1(x,y)$, (13) and the fact $B_- = B$ would imply the existence of $y'$ for which $B(y') \to I_1(x,y') = I_1(x,y') < I_1(x,y)$. But since $I_1(x,y) < a$, we have $I_1(x,y') < a$, hence also $I_1^+(x,y') = I_1(x,y')$ which would imply

$$A(x) = B^{\downarrow_{I_1^+}}(x) \leq B(y') \to I_1^+(x,y') = B(y') \to I_1(x,y') < I_1(x,y) = A(x),$$

a contradiction. Now, (14), (13), and the fact that for $x \notin {}^1A$ we have $A_-(x) = A(x)$ imply

$$B_-^{\downarrow_{I_1}}(x) = A(x) = A_-(x).$$

Second, let $x \in {}^1A$: Since $B_- = B$, we have

$$B_-^{\downarrow_{I_1}}(x) = \bigwedge_{y \notin {}^1B} (B(y) \to I_1(x,y)) \wedge \bigwedge_{y \in {}^1B} (B(y) \to I_1(x,y)).$$

Let us first observe that the assumption $B^{\downarrow_{I_1^+}}(x) = A(x) = 1$ and the fact that $b \to c = 1$ is equivalent to $b \leq c$ imply that $B(y) \leq I_1^+(x,y)$ for every $y \in Y$. Next, let us verify $\bigwedge_{y \notin {}^1B} B(y) \to I_1(x,y) = 1$. If $I_1^+(x,y) < 1$ then $I_1(x,y) =$

$I_1^+(x,y)$ and since $B(y) \leq I_1^+(x,y)$, we have $B(y) \leq I_1(x,y)$; if $I_1^+(x,y) = 1$, i.e. $I_1(x,y) = a$, then since $B(y) < 1$ by assumption, we have $B(y) < a = I_1(x,y)$, because $B(y)$ is equal to some $I_1^+(x',y') < 1$ and due to the ordinal equivalence of $I_1$ with $I_2$, all such values $I_1^+(x',y')$ are strictly smaller than $a$. Therefore, $B(y) \to I_1(x,y) = 1$ again. To sum up, $\bigwedge_{y \notin {}^1B} B(y) \to I_1(x,y) = 1$.

To verify $\bigwedge_{y \in {}^1B} B(y) \to I_1(x,y) = A_-(x)$, observe that for every $y \in {}^1B$, $B(y) \leq I_1^+(x,y)$ implies $I_1^+(x,y) = 1$, whence $I_1(x,y) = a$. Therefore, since ${}^1B \neq \emptyset$, there exists at least one $y \in {}^1B$, hence

$$\bigwedge_{y \in {}^1B} (B(y) \to I_1(x,y)) = \bigwedge_{y \in {}^1B} (B(y) \to a) = 1 \to a = a = A_-(x),$$

As a result, $B_-^{\downarrow_{I_1}}(x) = A_-(x)$, finishing the proof of $B_-^{\downarrow_{I_1}} = A_-$.

(3): By contradiction, assume ${}^1A = \emptyset = {}^1B$. From ${}^1A = \emptyset$ we get that for each $x \in X$ we have $A(x) = \bigwedge_{y \in Y} B(y) \to I_1^+(x,y) < 1$. Since $Y$ is finite, there exists $y \in Y$ such that $A(x) = B(y) \to I_1^+(x,y)$ and from the properties of $\to$ it follows that $A(x) = I_1^+(x,y) < B(y)$. Analogously, from ${}^1B = \emptyset$ we get that for each $y \in Y$ there is $x \in X$ with $B(y) < A(x)$. Now denote $n = \min(|X|,|Y|)$. If $|X| \leq |Y|$, take an arbitrary $x_1 \in X$. Due to the above observation, there is $y_1 \in Y$ with $A(x_1) < B(y_1)$. For $y_1$, there is $x_2 \in X$ with $B(y_1) < A(x_2)$. Repeating this argument we get some $y_n$ for which there should exist $x_{n+1} \in X$ such that $B(y_n) < A(x_{n+1})$. We obtained

$$A(x_1) < B(y_1) < A(x_2) < B(y_2) < \cdots < A(x_n) < B(y_n) < A(x_{n+1})$$

which is impossible since $X$ has exactly $n$ elements. □

**Lemma 3.** *The mapping $g$ defined by*

$$g(A,B) = \langle f^+_{I_1,I_1^+}(A), f^+_{I_1,I_1^+}(B) \rangle$$

*is a complete homomorphism of $\mathcal{B}(X,Y,I_1)$ onto $\mathcal{B}(X,Y,I_1^+)$ for which $g^{-1}(C,D)$ is a singleton or a two-element interval for each $\langle C,D \rangle \in \mathcal{B}(X,Y,I_1^+)$; in particular:*

$$g^{-1}(C,D) = \begin{cases} \{\langle C_-, D_- \rangle, \langle C^-, D^- \rangle\} & \text{if } {}^1C \neq \emptyset \text{ and } {}^1D \neq \emptyset, \\ \{\langle C, D \rangle\} & \text{otherwise.} \end{cases}$$

*Moreover, if $g(A,B) = \langle C,D \rangle$ then $A \equiv C$ and $B \equiv D$.*

*Proof.* Let for the element $a$ from (5), $h : L \to L$ be defined by

$$h(b) = \begin{cases} 1 & \text{if } b \geq a, \\ b & \text{if } b < a. \end{cases}$$

Clearly, $h$ coincides with $f^+_{I_1,I_1^+}$ on $I_1(X,Y) \cup \{1\}$, hence $g(A,B) = \langle h(A), h(B) \rangle$. One can easily observe that $h$ is a $\bigwedge$-morphism, i.e. a morphism that preserves arbitrary infima, of the residuated lattice $\mathbf{L}$ in $\mathbf{L}$. Furthermore, $h(I_1) = I_1^+$ since whenever $I_1(x,y) \geq a$, then $I_1(x,y) = a$ and thus $I_1^+(x,y) = 1 = h(a) = h(I_1(x,y))$; and if $I_1(x,y) < a$ then $h(I_1(x,y)) = I_1(x,y) = I_1^+(x,y)$. According to Theorem 3.2 of [3], $g$ is a complete homomorphism of $\mathcal{B}(X,Y,I_1)$ onto $\mathcal{B}(X,Y,I_1^+)$.

Let $^1C \neq \emptyset$ and $^1D \neq \emptyset$. We need to show that the set $g^{-1}(C, D)$ equals $\{\langle C_-, D_-\rangle, \langle C^-, D^-\rangle\}$. Clearly, due to Lemma 2 and the definition of $h$, we have $\langle C_-, D_-\rangle, \langle C^-, D^-\rangle \in g^{-1}(C, D)$. Furthermore, $\langle C_-, D_-\rangle$ is the least element of $g^{-1}(C, D)$. Namely, if $g(A, B) = \langle C, D\rangle$, we have for any $x \in X$ the following two possibilities. Either $C(x) < 1$ in which case $C(x) < a$, because $C(x)$ attains only values in $I_1^+(X, Y)$ and the largest one below 1 is strictly smaller than $a$, from which it follows that $A(x) = h(A(x)) = C(x) = C_-(x)$; or $C(x) = 1$ from which it follows that $A(x) \geq a = C_-(x)$. As a result, $C_- \subseteq A$, whence $\langle C_-, D_-\rangle \leq \langle A, B\rangle$. In a similar manner we get that $\langle C^-, D^-\rangle$ is the largest element of $g^{-1}(C, D)$. Therefore, it now suffices to show that there is no $\langle A, B\rangle \in \mathcal{B}(X, Y, I_1)$ for which $\langle C_-, D_-\rangle < \langle A, B\rangle < \langle C^-, D^-\rangle$. If this were the case, we would have $A(x_1) = 1$ and $A(x_2) = a$ for some $x_1, x_2$ for which $C(x_1) = C(x_2) = 1$ and $B(y_1) = 1$ and $B(y_2) = a$ for some $y_1, y_2$ for which $D(x_1) = D(x_2) = 1$. But this is impossible since then $1 = B(y_1) = \bigwedge_{x \in X} A(x) \to I_1(x, y_1)$ from which we get $A(x_1) \to I_1(x_1, y_1) = 1$, i.e. $A(x_1) \leq I_1(x_1, y_1)$. Since $A(x_1) = 1$, we get $I_1(x_1, y_1) = 1$, a contradiction to the fact that $I_1(x, y) \leq a$ for every $x$ and $y$.

Let $^1C = \emptyset$. We need to show that $g^{-1}(C, D) = \{\langle C, D\rangle\}$ in this case. Lemma 2 (3) implies that $^1D \neq \emptyset$. Clearly, we have $\langle C_-, D_-\rangle = \langle C, D\rangle$ and due to Lemma 2 (2), $\langle C, D\rangle = \langle C_-, D_-\rangle \in g^{-1}(C, D)$. Observe now that since $^1C = \emptyset$, $C$ is the only fuzzy set $A$ for which $h(A) = C$. As a result, $\langle C, D\rangle$ is the only element of $g^{-1}(C, D)$. If $^1D = \emptyset$, we proceed analogously and obtain $g^{-1}(C, D) = \{\langle C, D\rangle\}$.

The last claim to prove, i.e. that $g(A, B) = \langle C, D\rangle$ implies $A \equiv C$ and $B \equiv D$, is immediate. □

The following is a counterpart of Theorem 1 for the assumption of $I_1 \equiv I_2$ but not $I_1 \equiv_{\{1\}} I_2$. Without loss of generality we assume that $I_2(x, y) = 1$ for some $x$ and $y$. Let for the corresponding $f^+_{I_1,I_2}$ and $\langle C, D\rangle \in \mathcal{B}(X, Y, I_2)$ define the fuzzy sets $C_f, C^f \in L^X$ and $D_f, D^f \in L^Y$ by

$$C_f(x) = \min(f^+_{I_1,I_2})^{-1}(C(x)), \quad D_f(y) = \max(f^+_{I_1,I_2})^{-1}(D(y))$$

and

$$C^f(x) = \max(f^+_{I_1,I_2})^{-1}(C(x)), \quad D^f(y) = \min(f^+_{I_1,I_2})^{-1}(D(y)),$$

for every $x \in X$ and $y \in Y$, where $\min(f^+_{I_1,I_2})^{-1}(C(x))$ is the smallest $a \in L$ for which $f^+_{I_1,I_2}(a) = C(x)$ and analogously for the other cases. Observe that if $I_2 = I_1^+$ then $C_f = C_-$, $D_f = D_-$, $C^f = C^-$, and $D^f = D^-$, cf. (7) and (8).

**Theorem 3.** *Let $I_1 \equiv I_2$ but not $I_1 \equiv_{\{1\}} I_2$, let $I_2(x, y) = 1$ for some $x$ and $y$. Then the mapping $g$ defined by*

$$g(A, B) = \langle f^+_{I_1,I_2}(A), f^+_{I_1,I_2}(B)\rangle$$

*is a complete homomorphism of $\mathcal{B}(X, Y, I_1)$ onto $\mathcal{B}(X, Y, I_2)$ for which $g^{-1}(C, D)$ is a singleton or a two-element interval in $\mathcal{B}(X, Y, I_1)$ for each $\langle C, D\rangle \in \mathcal{B}(X, Y, I_2)$;*

*in particular:*

$$g^{-1}(C,D) = \begin{cases} \{\langle C_f, D_f\rangle, \langle C^f, D^f\rangle\} & \text{if } {}^1C \neq \emptyset \text{ and } {}^1D \neq \emptyset, \\ \{\langle C_f, D_f\rangle\} = \{\langle C^f, D^f\rangle\} & \text{otherwise.} \end{cases}$$

*Moreover, if* $g(A,B) = \langle C, D\rangle$ *then* $A \equiv C$ *and* $B \equiv D$.

*Proof.* The claim follows from Lemma 3 and Theorem 1. Namely, consider $I_1^+$ and the homomorphism $g_1 : \mathcal{B}(X,Y,I_1) \to \mathcal{B}(X,Y,I_1^+)$ from Lemma 3. Since $I_1^+$ clearly satisfies $I_1^+ \equiv_{\{1\}} I_2$, Theorem 1 implies the existence of an isomorphism $g_2 : \mathcal{B}(X,Y,I_1^+) \to \mathcal{B}(X,Y,I_2)$. The composition $g$ of $g_1$ and $g_2$ satisfies the required properties, which is an easy consequence of Lemma 3; Theorem 1; the fact that ${}^1C \neq \emptyset$ if and only if ${}^1(f^{-1}_{I_1^+,I_2} \circ C) \neq \emptyset$ and the same for $D$; and due to

$$\langle C_f, D_f\rangle = \langle (f^{-1}_{I_1^+,I_2} \circ C)_-, (f^{-1}_{I_1^+,I_2} \circ D)_-\rangle$$

and

$$\langle C^f, D^f\rangle = \langle (f^{-1}_{I_1^+,I_2} \circ C)^-, (f^{-1}_{I_1^+,I_2} \circ D)^-\rangle.$$

□

It is easy to see that Theorem 1 and Theorem 3 may be brought into one theorem assuming the weaker condition $I_1 \equiv I_2$ and handling both cases, $I_1 \equiv_{\{1\}} I_2$ and not $I_1 \equiv_{\{1\}} I_2$, because if $I_1 \equiv_{\{1\}} I_2$ then as one easily checks, $\langle C_f, D_f\rangle = \langle C^f, D^f\rangle$.

**Remark 1.** The homomorphism $g$ from Theorem 3 may be considered an "almost isomorphism" because only certain concepts of $\mathcal{B}(X,Y,I_2)$ have non-singleton preimages and these are two-element intervals. Those intervals consist of two very similar formal concepts because one is brought to the other by switching 1s and $a$s, where $a$ is a truth degree smaller than 1 but larger than any other truth degree involved in these formal concepts. Moreover, it is easy to see that the mapping sending each $\langle C, D\rangle$ to $\langle C_f, D_f\rangle$ as well as the one sending each $\langle C, D\rangle$ to $\langle C^f, D^f\rangle$ are order embeddings of $\mathcal{B}(X,Y,I_2)$ to $\mathcal{B}(X,Y,I_1)$.

**Example 4.** In this example, we illustrate Theorem 3, as well as Lemma 2 and Lemma 3. Consider the fuzzy relations $I$, $J$, and $K$ from Example 1 (see also Example 3). Put $I_1 = I$ and $I_2 = K$. As $I \equiv K$, $I \not\equiv_{\{1\}} K$, and $1 \notin I(X,Y)$, Lemma 2, Lemma 3 and Theorem 3 apply. Notice first that in (5), we have $a = 3/4$, $x = x_1$, and $y = y_1$. Therefore, the relation $I_1^+$ defined by (6) coincides with $J$. For the mapping $f^+_{I_1,I_1^+} = f^+_{I,J}$ of $I(X,Y) \cup \{1\}$ to $J(X,Y) \cup \{1\}$ we have

$$f^+_{I_1,I_1^+}(0) = 0,\ f^+_{I_1,I_1^+}(1/4) = 1/4,\ f^+_{I_1,I_1^+}(3/4) = 1, \text{ and } f^+_{I_1,I_1^+}(1) = 1.$$

According to Lemma 3, $f^+_{I_1,I_1^+}$ induces a complete onto homomorphism, denoted here $g_1$, of $\mathcal{B}(X,Y,I_1)$ into $\mathcal{B}(X,Y,I_1^+)$. The Hasse diagrams of $\mathcal{B}(X,Y,I_1)$ and $\mathcal{B}(X,Y,I_1^+)$ along with $g_1$ are depicted in the left part of Fig. 1. Since $I_1^+ = J$ and $I_2 = K$, Example 3 tells us that there exists an isomorphism of $\mathcal{B}(X,Y,I_1^+)$ onto $\mathcal{B}(X,Y,I_2)$, denoted here $g_2$. This isomorphism is depicted in the right part

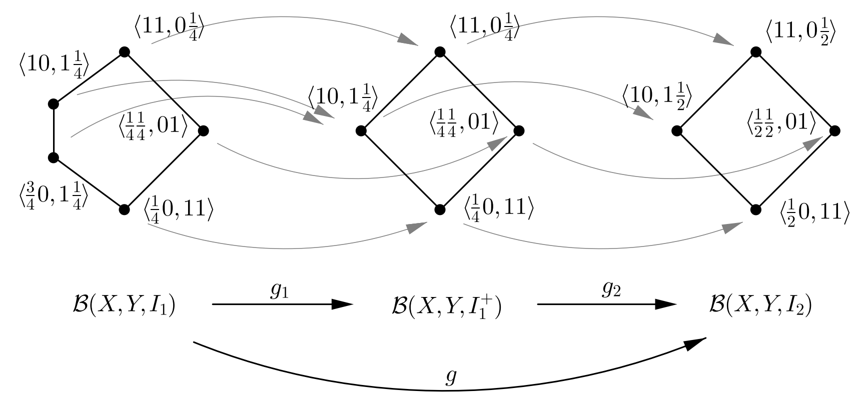


Figure 1: Concept lattices and homomorphisms from Example 4.

of Fig. 1. The complete homomorphism $g$ of $\mathcal{B}(X, Y, I_1)$ onto $\mathcal{B}(X, Y, I_2)$ from Theorem 3 results as the composition of $g_1$ and $g_2$ (cf. also the proof of Theorem 3). Notice that according to Theorem 3, the formal concept $\langle C, D\rangle = \langle 10, 1\frac{1}{2}\rangle$ of $\mathcal{B}(X, Y, I_2)$ has two preimages, $\langle C_f, D_f\rangle = \langle \frac{3}{4}0, 1\frac{1}{4}\rangle$ and $\langle C^f, D^f\rangle = \langle 10, \frac{3}{4}\frac{1}{4}\rangle$, while every other formal concept in $\mathcal{B}(X, Y, I_2)$ has a single preimage.

## 4. Conclusions

We proved that if two data tables with fuzzy attributes are ordinally equivalent, i.e. one may be brought to the other by means of an increasing function, the associated concept lattices based on Gödel fuzzy logic connectives are almost isomorphic (cf. Remark 1) and consist of ordinally equivalent formal concepts. If, moreover, the tables agree on entries with degree 1, representing that the attribute fully applies to the object, the concept lattices are isomorphic. We also showed that the assumption of Gödel operations is essential. The results confirm the experience of practitioners using fuzzy logic, sometimes articulated in an informal manner, that with Gödel connectives, what matters is the ordering of the truth degrees involved. This paper illustrates that such intuition, as well as further issues related to the general question of the significance of values of truth degrees, may properly be addressed from the standpoint of measurement theory. The paper suggests that from the practical viewpoint, measurement-theoretic-like results may provide a guide to the choice of fuzzy logic connectives. In case of the results presented in this paper, a user is told to use Gödel operations if the prospect of ordinally equivalent data leading to isomorphic concept lattices is appealing or required. In this perspective, one stream of possible future research includes the investigation of a similar kind of results regarding further fuzzy logic connectives in more general settings of formal concept analysis such as those proposed in [6, 18, 21], as well as in fuzzy logic modeling of concepts in general [7, 8]. In a broader perspective, examination of the problems addressed in this paper in a broader context of fuzzy logic modeling seems a much needed project.

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