Derivation digraphs for graded if-then rules

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Overview

1. Preliminaries
2. Graded If-then formulas in FCA of data with fuzzy attributes and in similarity-based relational model
3. Graph-based method of reasoning with graded if-then formulas

Preliminaries: structures of truth degrees

Complete residuated lattice: $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1 \rangle$:

Truth-stressing hedges (Takeuti+Titani, Baaz, Hájek,...)
idempotent truth-stressing **hedge**
mapping $^* \colon L \to L$ satisfying

$$1^* = 1, \qquad a^* \leq a, \qquad (a \to b)^* \leq a^* \to b^*, \qquad a^{**} = a^*,$$

meaning of *: truth function of logical connective "very true"

Two boundary hedges

1. **identity**, i.e. $a^* = a$ $(a \in L)$;
2. **globalization**: $a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$ ... interp. "fully true"

Notes on extension of the Codd model

- Y: set of all attributes
- Relation Schemes = subsets $R \subseteq Y$
- For $R \subseteq Y$ and domains D_y: $\mathrm{Tupl}(R) = \prod_{y \in R} D_y$
- **Similarity relation**: each domain D_y equipped with binary $\mathbf{L}$-relation $\approx_y$ satisfying:

(Ref) for each $u \in D_y$: $u \approx_y u = 1$,
(Sym) for each $u, v \in D_y$: $u \approx_y v = v \approx_y u$.

- **Domain with similarity** = $\langle D_y, \approx_y \rangle$

Definition (Ranked Data Table (RDT))

Let $R \subseteq Y$ be a relation scheme and $\langle D_y, \approx_y \rangle$ be a domain with similarity (for each $y \in R$). A **ranked data table on R over $\{\langle D_y, \approx_y \rangle \mid y \in R\}$** is any finite $\mathbf{L}$-set in $\mathrm{Tupl}(R)$.

Notes on extension of the Codd model: continue

Example

Similarity-based query: "Show me all cars which are sold for (approximately) $7\,000$ € and are (approximately) 3 years old."

brand	*model*	*price*	*year*	*km*
Renault	Scénic	$6\,940$	2009	$115\,556$
Renault	Scénic	$7\,200$	2010	$101\,478$
Opel	Zafira	$7\,500$	2008	$130\,656$
Citroen	Picasso	$7\,925$	2009	$109\,015$
Volkswagen	Caddy	$8\,600$	2008	$122\,855$

Notes on extension of the Codd model: continue

Example

Similarity-based query: "Show me all cars which are sold for (approximately) $7\,000$ € and are (approximately) 3 years old."

$\mathcal{D}(t)$
0.97
0.8
0.75
0.54
0.1

brand	*model*	*price*	*year*	*km*
Renault	Scénic	$6\,940$	2009	$115\,556$
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Fuzzy Attribute Implication (FAI)

Y - finite nonempty set of attributes (features)

Definition

Fuzzy attribute implication (graded if-then rule) on Y is any expression of the from $A \Rightarrow B$, where $A, B \in L^Y$. An $\mathbf{L}$-set T of FAIs on Y shall be called a **theory**.

Two basic interpretations:

1. attribute implications in FCA of data with fuzzy attributes
   - For every object: "If it is very true that an object has all attributes from A, then it has also all attributes from B."
2. similarity-based functional dependencies in an extension of the Codd model of data
   - For every pair of tuples: "If they have very similar values on attributes from A, then they have similar values on attributes from B."

Semantic entailment

1. Given a fuzzy attribute implication $A \Rightarrow B$:
   - $M \in L^Y$, $||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M)$
   - $||A \Rightarrow B||_{\mathcal{D}} = \bigwedge_{t_1, t_2 \in \mathrm{Tupl}(R)} \Big((t_1(A) \approx_{\mathcal{D}} t_2(A))^* \rightarrow (t_1(B) \approx_D t_2(B))\Big)$
     $t_1(C) \approx_{\mathcal{D}} t_2(C) = (\mathcal{D}(t_1) \otimes \mathcal{D}(t_2)) \rightarrow \bigwedge_{y \in R}(C(y) \rightarrow t_1(y) \approx_y t_2(y))$
2. **Model** of theory T (fuzzy set of FAIs):
   - $\mathrm{Mod}^{AI}(T) = \{M \in L^Y \,|$ for each $A, B \in L^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}$
   - $\mathrm{Mod}^{FD}(T) = \{\mathcal{D}|$ for each $A, B \in L^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_D\}$
3. A degree $||A \Rightarrow B||_T$ to which $A \Rightarrow B$ **semantically follows** from T:
   - $||A \Rightarrow B||_T^{AI} = \bigwedge_{M \in \mathrm{Mod}^{AI}(T)} ||A \Rightarrow B||_M$
   - $||A \Rightarrow B||_T^{FD} = \bigwedge_{\mathcal{D} \in \mathrm{Mod}^{FD}(T)} ||A \Rightarrow B||_{\mathcal{D}}$

Theorem (Bělohlávek, Vychodil, 2005)

For any fuzzy set T of FAIs and any FAI $A \Rightarrow B$ we have: $||A \Rightarrow B||_T^{AI} = ||A \Rightarrow B||_T^{FD}$

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Armstrong-like axiomatic system

$\mathbf{L}$ finite, Y set of attributes, $A, B, C, D \in L^Y$:

(Ax) infer $A \cup B \Rightarrow B$

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$

(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$,

Two kind of completeness can be shown for all finite residuated lattices:

- **Ordinary** completeness
  - T - crisp theory
  - $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T (in degree 1)
- **Graded (Pavelka-style)** completeness
  - T - an $\mathbf{L}$-set
  - A degree to which $A \Rightarrow B$ semantically follows from T equals to degree to which $A \Rightarrow B$ is provable from T

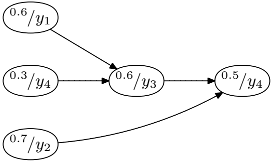
$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid T \vdash A \Rightarrow c \otimes B\}$$

Bělohlávek, Vychodil, 2006: Completeness for general complete residuated lattices (+ infinitary deduction rule)

Construction of derivation digraphs for FAIs

David Maier (1980): graph-based approach - to normalize proof. Can this approach be extended for FAIs? **YES**.

Derivation digraphs:



- digraph - directed graph
- acyclic graph
- vertices are attributes from Y
- arcs correspond to FAIs used from an input theory

Construction of T-based $\mathbf{L}^*$-derivation DAG: T be a set of FAIs over Y.

- Any $\mathbf{D} = \langle V, \emptyset \rangle$ such that $\emptyset \neq V \subseteq Y \times L$ and for every $y \in Y$ there is at most one $a \in L$ such that $\langle y, a \rangle \in V$, is a T-based $\mathbf{L}^*$-derivation DAG;

Construction of derivation digraphs for FAIs: continue

- New vertex for attribute y

  If $\mathbf{D} = \langle V, A \rangle$ is a T-based $\mathbf{L}^*$-derivation DAG and there are $E \Rightarrow F \in T$, attribute $y \in Y$, and vertices $\langle y_1, a_1 \rangle \in V, \ldots, \langle y_k, a_k \rangle \in V$ such that for

$$s_0 = \bigwedge\{E(y) \to 0 \,|\, y \in Y \text{ and } y \notin \{y_1, \ldots, y_k\}\},$$
$$s_1 = \bigwedge\{E(y_i) \to a_i \,|\, i = 1, \ldots, k\},$$
$$m = \bigvee\{a \in L \,|\, \langle y, a \rangle \in V\},$$
$$d = \big((s_0 \wedge s_1)^* \otimes F(y)\big) \vee m,$$

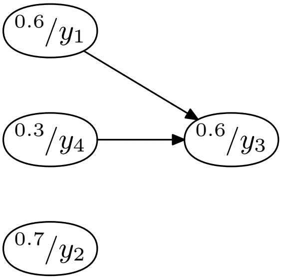
  we have $d > m$, then $\mathbf{D}' = \langle V', A' \rangle$, where

$$V' = V \cup \{\langle y, d \rangle\},$$
$$A' = A \cup \{\langle \langle y_i, a_i \rangle, \langle y, d \rangle \rangle \,|\, i = 1, \ldots, k\},$$

  is a T-based $\mathbf{L}^*$-derivation DAG.

One step in construction of DAG

- $\mathbf{L}^*$: finite linearly ordered Łukasiewicz algebra with $L = \{0, 0.1, \ldots, 0.9, 1\} \subseteq [0, 1]$ $\wedge$ and $\vee$ usual minima and maxima, $\otimes, \rightarrow$ Łukasiewicz operations, $a^* = 1$ for $a = 1$, $a^* = 0.6$ for $0.6 \le a \le 0.9$, $a^* = 0.2$ for $0.2 \le a \le 0.5$, $a^* = 0$ for $0 \le a \le 0.1$
- set of initial vertices: $\{{}^{0.6}/y_1, {}^{0.7}/y_2, {}^{0.3}/y_4\}$, theory: $T = \{\ldots, \{{}^{0.7}/y_2, {}^{0.9}/y_3\} \Rightarrow \{{}^{0.9}/y_4\}, \ldots\}$
- adding vertex $\langle y_4, 0.5 \rangle$



$s_0 = 1$
$s_1 = 0.7$
$m = 0.3$
$d = (0.7^* \otimes 0.9) \vee 0.3$
$= (0.6 \otimes 0.9) \vee 0.3$
$= 0.5$

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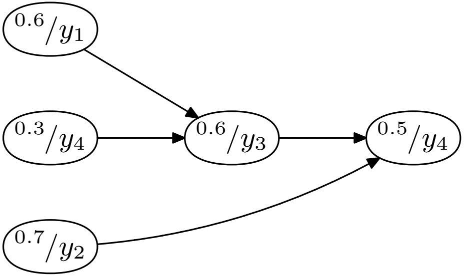
- adding vertex $\langle y_4, 0.5\rangle$



$$
\begin{aligned}
s_0 &= 1 \\
s_1 &= 0.7 \\
m &= 0.3 \\
d &= (0.7^* \otimes 0.9) \vee 0.3 \\
&= (0.6 \otimes 0.9) \vee 0.3 \\
&= 0.5
\end{aligned}
$$

Derivation DAG: further notions

For T-based $\mathbf{L}^*$-derivation DAG $\mathbf{D}$ we have:

- **Yield of $\mathbf{D}$ on attribute y**: $\mathbf{D}(y) = \bigvee\{a \in L \mid \langle y, a\rangle \in V\}$
- $\mathbf{D}$ is called a **T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$** if
  1. $\mathbf{D}(y) \geq B(y)$ for all $y \in Y$.
  2. If $A \neq \emptyset$, then the set of initial vertices of $\mathbf{D}$ is
  $$\{\langle y, A(y)\rangle \mid y \in Y \text{ and } A(y) > 0\}$$
  3. If $A = \emptyset$, then the set of initial vertices of $\mathbf{D}$ is $\{\langle y^{\#}, 0\rangle\}$, where $y^{\#}$ is a fixed attribute from Y.

Results: Completeness

Let T be a crisp theory

Theorem

If $T \vdash A \Rightarrow B$ (using (Ax),(Cut),(Mul)), then there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$.

Proof.

Assume that $T \vdash A \Rightarrow B$. Then, we can show (by induction) that there is a sequence $A_0, \ldots, A_n$ of $\mathbf{L}$-sets in Y such that $A_0 = A$, $B \subseteq A_n$ and for all $i = 0, \ldots, n-1$ there is some $E \Rightarrow F \in T$ and attribute $y \in Y$ such that

$$A_{i+1} = A_i \cup \left(S(E, A_i)^* \otimes \left\{{}^{F(y)}\!/y\right\}\right)$$

$A_i \subset A_{i+1}$, and $T \vdash A \Rightarrow A_i$ for all $i = 0, \ldots, n$. Based on this sequence we will construct T-based $\mathbf{L}^*$-derivation DAGs $\mathbf{D}_0, \ldots, \mathbf{D}_n$. $\mathbf{D}_0$ contains just the initial vertices corresponding to A and $\mathbf{D}_{i+1}$ results from $\mathbf{D}_i$ by adding the vertex $\langle y, A_{i+1}(y)\rangle$ where $A_{i+1}(y) = A_i(y) \vee (S(E, A_i)^* \otimes F(y))$. Obviously, for each $y \in Y$, we have $\mathbf{D}_i(y) = A_i(y)$. In a particular case, $\mathbf{D}_n$ is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$. □

Theorem

If there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$, then $T \vdash A \Rightarrow B$.

Proof.

By induction on the complexity of T-based $\mathbf{L}^*$-derivation DAGs:
1) If $\mathbf{D}$ consists solely of the initial vertices, then $T \vdash A \Rightarrow B$ follows using (Ax).
2) Assume that $\mathbf{D}$ results from $\mathbf{D}' = \langle V', A' \rangle$ by adding vertex $\langle y, d \rangle$. Therefore, there are $E \Rightarrow F \in T$, attribute $y \in Y$, and vertices $\langle y_1, a_1 \rangle \in V', \ldots, \langle y_k, a_k \rangle \in V'$ such that $\mathbf{D}$ results by adding vertex $\langle y, d \rangle$, where $d = ((s_0 \wedge s_1)^* \otimes F(y)) \vee \mathbf{D}'(y)$. Let $B_{\mathbf{D}'} \in L^Y$ such that $B_{\mathbf{D}'}(y) = \mathbf{D}(y)$, $B \subseteq \{{}^{d}\!/y\} \cup B_{\mathbf{D}'}$. Consider $G \subseteq B_{\mathbf{D}'}$ such that $G(y_i) = a_i$ (for all $i = 1, \ldots, k$) and $G(y') = 0$ otherwise. It can be shown that from induction hypothesis $T \vdash A \Rightarrow G$ follow also $T \vdash A \Rightarrow \{{}^{S(E,G)^* \otimes F(y)}\!/y\} \cup B_{\mathbf{D}'}$. Since $s_0 \wedge s_1 = S(E, G)$, we have $T \vdash A \Rightarrow \{{}^{d}\!/y\} \cup B_{\mathbf{D}'}$ and by derived inference rule Projectivity $T \vdash A \Rightarrow B$. □

Theorem

If $\mathbf{L}$ is finite, then $A \Rightarrow B$ semantically follows from theory T in degree 1 iff there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow B$.

Proof.

Follows from previous theorems and completeness of fuzzy attribute logic. □

Theorem

If $\mathbf{L}$ is finite, then $||A \Rightarrow B||_T$ is the greatest degree $a \in L$ such that there is a T-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow a{\otimes}B$.

Proof.

Follow from the fact that

$$||A \Rightarrow B||_T = \bigvee\{c \in L \,|\, ||A \Rightarrow c{\otimes}B||_T = 1\}$$

and from previous theorem. □

Results: Completeness

Let T be an $\mathbf{L}$-set
We can define crisp(T) by:

$$\text{crisp}(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B | A, B \in L^Y \text{ and } T(A \Rightarrow B) \otimes B \not\subseteq A\}$$

We know:

- $\text{Mod}(T) = \text{Mod}(\text{crisp}(T))$
- $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\text{crisp}(T)}$.

Theorem

If $\mathbf{L}$ is finite and T is an $\mathbf{L}$-set of FAIs, then $||A \Rightarrow B||_T$ is the greatest degree $a \in L$ such that there is a T'-based $\mathbf{L}^*$-derivation DAG for $A \Rightarrow a \otimes B$, where $T' = \text{crisp}(T)$.

Results: determine degree $||A \Rightarrow B||_T$

Main ideas ($\mathbf{L}$, Y finite):

- A_T^+: largest $\mathbf{L}$-set such that $T \vdash A \Rightarrow A_T^+$
- Property (Bělohlávek, Vychodil, 2006): $||A \Rightarrow B||_T = S(B, A_T^+)$
- Informally: Final T-based $\mathbf{L}^*$-derivation DAG - cannot be enlarged

Theorem

Let $\mathbf{D}$ be a derivation DAG for $A \Rightarrow B$, then $\mathbf{D}$ is final iff $\mathbf{D}(y) = A_T^+(y)$ for all $y \in Y$

1. Construct a T-based $\mathbf{L}^*$-derivation DAG $\mathbf{D} = \langle V, \emptyset \rangle$ with
$$V = \{\langle y, A(y)\rangle \mid A(y) > 0\};$$
If $V = \emptyset$, put $V = \{\langle y^\sharp, 0\rangle\}$.
2. If $\mathbf{D}$ can be enlarged according to Definition (case 2.), then enlarge $\mathbf{D}$ and repeat step 2.; otherwise return $S(B, A_T^+)$, where $A_T^+(y) = \mathbf{D}(y)$ for all $y \in Y$.

Summary

- graph-based inference for FAIs
- degrees of semantic entailment of FAIs from a theory can be characterized by the existence of directed acyclic graphs
- characterization of syntactic closures via derivation DAGs
- algorithm for computing closures

Future work:

- further properties of derivation DAGs, complexity issues