Attribute Dependencies in a Fuzzy Setting

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OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Outline

1. Attribute Dependencies
2. Straightforward Generalisation
3. Attribute Dependencies in a Fuzzy Setting

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	genus				habitat				size			fur		color			
	Acinonyx	Felis	Leptailurus	Panthera	Africa	America	Asia	Europe	small	medium	large	stripes	spots	black	sandy	white	yellow
Cheetah	×				×					×		×	×				×
Cougar				×		×					×				×		
Jaguar				×		×					×		×	×			×
Lion				×	×						×				×		
Panther				×	×		×			×			×				×
Serval			×		×				×			×	×		×		×
Tiger				×			×				×	×				×	×
Wildcat		×			×	×	×	×	×			×	×	×	×		×

- 35 concepts
- Biologist: What are "felines with yellow fur" ?
- Biologist's order on the attribute
  - fur pattern and color are **less important than** size
  - size is **less important than** genus
  - size is **less important than** habitat
- $\Longrightarrow$ 15 " interesting" concepts.

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Definition (Belohlavek, Sklenar, Vychodil)

An **attribute dependency formula (AD formula)** over a set Y is

$$A \sqsubseteq B;$$

$A; B \subseteq Y$. $A \sqsubseteq B$ is **true** in $M \subseteq Y$, written $M \models A \sqsubseteq B$, if

$$\text{if } A \cap M \neq \emptyset; \text{ then } B \cap M \neq \emptyset:$$

$(C; D) \in \mathcal{B}(X; Y; I)$ satisfies $A \sqsubseteq B$ if

$$D \models A \sqsubseteq B:$$

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$$D \models A \sqsubseteq B:$$

$A := \{\text{black, sandy, white, yellow, stripes, spots}\} \sqsubseteq \{\text{ small, medium, large }\} =: B$,

$(C; D) := (\{\text{cheetah, jaguar, panther, serval, tiger, wildcat}\}; \{\text{ yellow}\});$

$A \cap D = \{\text{yellow}\}$ but $B \cap D = \varnothing$

$D \not\models A \sqsubseteq B$

Definition

$M \subseteq Y$ **model** of a set T of AD formulas if $M \models A \sqsubseteq B$, for every $A \sqsubseteq B \in T$.

$$Mod(T) := \{M \subseteq Y \mid M \models A \sqsubseteq B; \text{ for every } A \sqsubseteq B \in T\}:$$

Theorem (Belohlavek, Vychodil)

$Mod(T)$ is a kernel system.

Lemma (Belohlavek, Vychodil)

For $A; B; M \subseteq Y$, we have

$$M \models A \sqsubseteq B \Longleftrightarrow \overline{M} \models B \Rightarrow A:$$

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Generation of minimal bases

- associate to each T a closure operator C
- use NEXTCLOSURE to compute minimal base T' (of attribute implications) associated to C
- $T_{\min} := \{B \setminus A \sqsubseteq A \mid A \Rightarrow B \in T'\}$

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$A, B \in \mathbf{L}^Y$, a **fuzzy attribute dependency formula (fAD)**

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$A; B; C; D; M \in \mathbf{L}^Y$

- $\in L \setminus \{0\}$, $C \cap D$ **-true** if

$$\exists y \in Y \text{ such that } (C \cap D)(y) \geq$$

- $;\ \in L \setminus \{0\}$, $\leq$, $M \models_{\alpha,\beta} A \sqsubseteq B$ if

$$\text{if } A \cap M \quad \text{-true, then } B \cap M \quad \text{-true;}$$

- T set of fuzzy AD formulas,

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Fuzzy Kernel Operators and Systems [Belohlavek, Funiokova, Vychodil]

- An $\mathbf{L}^*$**-kernel operator** on a set Y is a mapping $\; : \mathbf{L}^Y \to \mathbf{L}^Y$ satisfying

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\begin{aligned}
(A) &\subseteq A; \\
S(A_1; A_2)^* &\leq S(\;(A_1);\;(A_2)); \\
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- A system $\mathcal{S} := \{A_j \in \mathbf{L}^Y \mid j \in J\}$ is a $\mathbf{L}^*$**-kernel system** if for each $A \in \mathbf{L}^U$ holds that

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\bigcup_{j \in J} S(A; A_j)^* \otimes A_j \in \mathcal{S}
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- $\mathcal{S}$ closed under arbitrary unions is a $\mathbf{L}^*$-kernel system iff for each $a \in L$ and $A \in \mathcal{S}$ it holds $a^* \otimes A \in \mathcal{S}$.

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Is $Mod\,(T)$ an $\mathbf{L}^*$-kernel system?

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For all $y \in Y$ holds

$$(a^* \otimes M)(y) = a^* \otimes M(y) = \begin{cases} 0; & a = 0 \\ M(y); & a = 1 \end{cases}$$

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	good team player		good organisational skills			adaptive towards new			confidential		computer skills	
	a: collaborative	b: not discriminative	c: time management	d: problem solver	e: analytical thinking	f: environment	g: assignment	h: priority	i: judgement	j: discretion	k: word processing	l: database
1	0	0.5	0.5	1	1	0	0	0.5	0.5	0.5	1	1
2	1	1	1	1	1	1	1	1	0.5	0.5	0	0
3	0.5	0.5	0.5	1	1	0	1	1	1	1	0.5	0.5
4	1	0.5	1	1	1	0	1	1	0.5	0.5	1	1
5	0	0.5	0	0.5	0.5	0	0	0.5	0.5	0.5	0	0
6	1	1	0.5	1	1	1	1	1	0.5	0.5	0.5	0.5
7	0	0.5	0	0	0.5	0	0	0.5	0	0.5	0	0

- Gödel logic $\Longrightarrow$ 44 fuzzy concepts
- $\{^{0.5}$=time, problem, analytical$\} \sqsubseteq$ {collaborative, not discriminative}
  $\{^{0.5}$=judgement, $^{0.5}$=discretion$\} \sqsubseteq$ {enviroment, assigments, priority}
  $\Longrightarrow$ 11 fuzzy concepts

Extent							Intent											
1	2	3	4	5	6	7	a	b	c	d	e	f	g	h	i	j	k	l
0	0	0	0	0	0.5	0	1	1	1	1	1	1	1	1	1	1	1	1
0	0.5	0	0	0	0.5	0	1	1	1	1	1	1	1	1	1	1	0	0
0	1	0	0	0	0.5	0	1	1	1	1	1	1	1	1	0.5	0.5	0	0
0	0	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0.5	0.5
0	1	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0	0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0	1	0	0	1	0	0	1	0	1	0	0
0.5	1	0.5	0.5	0.5	1	0.5	0	1	0	0	1	0	0	1	0	0.5	0	0
0.5	0.5	1	0.5	0.5	0.5	0.5	0	0.5	0	0	1	0	0	1	0	1	0	0
0.5	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	1	0	0.5	0	0
1	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	0.5	0	0.5	0	0
1	1	1	1	1	1	1	0	0.5	0	0	0.5	0	0	0.5	0	0.5	0	0

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0	0.5	0	0	0	0.5	0	1	1	1	1	1	1	1	1	1	1	0	0
0	1	0	0	0	0.5	0	1	1	1	1	1	1	1	1	0.5	0.5	0	0
0	0	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0.5	0.5
0	1	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0	0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0	1	0	0	1	0	0	1	0	1	0	0
0.5	1	0.5	0.5	0.5	1	0.5	0	1	0	0	1	0	0	1	0	0.5	0	0
0.5	0.5	1	0.5	0.5	0.5	0.5	0	0.5	0	0	1	0	0	1	0	1	0	0
0.5	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	1	0	0.5	0	0
1	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	0.5	0	0.5	0	0
1	1	1	1	1	1	1	0	0.5	0	0	0.5	0	0	0.5	0	0.5	0	0

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0	1	0	0	0	0.5	0	1	1	1	1	1	1	1	1	0.5	0.5	0	0
0	0	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0.5	0.5
0	1	0	0	0	1	0	1	1	0.5	1	1	1	1	1	0.5	0.5	0	0
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0	1	0	0	1	0	0	1	0	1	0	0
0.5	1	0.5	0.5	0.5	1	0.5	0	1	0	0	1	0	0	1	0	0.5	0	0
0.5	0.5	1	0.5	0.5	0.5	0.5	0	0.5	0	0	1	0	0	1	0	1	0	0
0.5	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	1	0	0.5	0	0
1	1	1	1	0.5	1	0.5	0	0.5	0	0	1	0	0	0.5	0	0.5	0	0
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0	1	0.5	0.5	0	0.5	0	1	1	1	1	1	0	1	1	0.5	0.5	0	0
0	0	0.5	0.5	0	1	0	1	1	0.5	1	1	0	1	1	0.5	0.5	0.5	0.5
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0	1	0.5	0.5	0	0.5	0	1	1	1	1	1	0	1	1	0.5	0.5	0	0
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where $A; B \in \mathbf{L}^Y$.

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A formal fuzzy concept $(C; D) \in \mathfrak{B}(X; Y; I)$ satisfies $A \sqsubseteq B$ if $D \models A \sqsubseteq B$.
T set of fAD formulas. Denote

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$\underline{\mathfrak{B}}_T(X; Y; I)$ **fuzzy concept lattice of** $(X; Y; I)$ **constrained by** T

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$$\mathfrak{B}_T(X; Y; I) := \{(C; D) \in \mathfrak{B}(X; Y; I) \mid D \models A \sqsubseteq B; \text{ for every } A \sqsubseteq B \in T\};$$

$$\underline{\mathfrak{B}}_T(X; Y; I) := (\mathfrak{B}_T(X; Y; I); \leq);$$

$\underline{\mathfrak{B}}_T(X; Y; I)$ **fuzzy concept lattice of** $(X; Y; I)$ **constrained by** T

Proposition

Let T be a set of fAD formulas. Then, $\underline{\mathfrak{B}}_T(X;Y;I)$ is a complete fuzzy lattice, which is a $\bigvee$-sublattice of $\underline{\mathfrak{B}}(X;Y;I)$.

- bounded from below
- arbitrary suprema
  $(A_j;B_j) \in \mathfrak{B}_T(X;Y;I) \Longrightarrow \cap_{j\in J} B_j \models A \sqsubseteq B$ for all $A \sqsubseteq B \in T$
- in general not a $\bigwedge$-sublattice of $\underline{\mathfrak{B}}(X;Y;I)$
  $\bigcup_{j\in J} B_j \subseteq (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}$
- Top elements of $\underline{\mathfrak{B}}_T(X;Y;I)$ and $\underline{\mathfrak{B}}(X;Y;I)$ are the same if $\nexists A \sqsubseteq B \in T$ such that every attribute from A is shared by all objects from X with at least the truth degree and there is at least one attribute from B which is not shared by all objects from X with at least the truth value .

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$$Str : \mathcal{P}(fADF) \to \mathcal{P}(\mathfrak{B}(X;Y;I));$$
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$$Str(T) := \{(A;B) \in \mathfrak{B}(X;Y;I) \mid (A;B) \models ' \text{ for each } ' \in T\};$$
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Definition

A set $M \in \mathbf{L}^Y$ is a **model** of a set T of fAD formulas if for each $A \sqsubseteq B \in T$, $M \models A \sqsubseteq B$.

$$Mod(T) := \{M \in \mathbf{L}^Y \mid M \models A \sqsubseteq B; \text{ for each } A \sqsubseteq B \in T\}.$$

An fAD formula $A \sqsubseteq B$ **follows semantically** from T, written $T \models A \sqsubseteq B$, if for each $M \in Mod(T)$, we have $M \models A \sqsubseteq B$.

Lemma

1. $M \models A \sqsubseteq \{^{l_1}{=}y_1; : : : ;^{l_m}{=}y_m\}$ *if and only if for each* $i \in \{1; : : : ; m\}$ *we have* $M \models A \sqsubseteq\ ^{l_i}{=}y_i$.
2. *For each set* T *of fAD formulas and each fAD formula* ' , $T \models$ ' *if and only if* $\lfloor T \rfloor \models$ ' , *where* $\lfloor T \rfloor := \{A \sqsubseteq\ ^{l}{=}z \mid A \sqsubseteq B \in T$ *and* $B(z) = l\}$.

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- An $\mathbf{L}^*$**-closure operator** on a set Y is a mapping $C : \mathbf{L}^Y \to \mathbf{L}^Y$ satisfying

$$A \subseteq C(A);$$
$$S(A_1; A_2)^* \leq S(C(A_1); C(A_2));$$
$$C(A) = C(C(A));$$

  for every $A; A_1; A_2 \in \mathbf{L}^Y$.

- A system $\mathcal{S} := \{A_j \in \mathbf{L}^Y \mid j \in J\}$ is an $\mathbf{L}^*$**-closure system** if for each $A \in \mathbf{L}^U$ holds that

$$\bigcap_{j \in J} S(A; A_j)^* \to A_j \in \mathcal{S}$$

- $\mathcal{S}$ closed under arbitrary intersections is an $\mathbf{L}^*$-closure system iff for each $a \in L$ and $A \in \mathcal{S}$ it holds $a^* \to A \in \mathcal{S}$.

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Let T be a set of fAD formulas. Then, $Mod(T)$ is an $\mathbf{L}^*$-closure system.

- $Mod(T)$ is closed under arbitrary intersections,
- if $M \in Mod(T)$, then $a^* \to M \in Mod(T)$ if and only if * is the globalisation, because

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Lemma

For any $\mathbf{L}^*$*-closure system* $\mathcal{S}$ *in* Y *there is a set* T *of fAD formulas over* Y *such that* $\mathcal{S} = Mod(T)$.

- $T := \{A \sqsubseteq C_{\mathcal{S}}(A) \mid A \in \mathbf{L}^Y\}, \quad = \quad = 1;$
- $M \in \mathcal{S}$
  if $S(A; M) < 1$, $M \models A \sqsubseteq C_{\mathcal{S}}(A)$ and we are done
  if $S(A; M) \geq 1 \Longrightarrow C_{\mathcal{S}}(A) \subseteq C_{\mathcal{S}}(M) = M \Longrightarrow S(C_{\mathcal{S}}(A); M) \geq 1$
- $M \in Mod(T)$, i.e.,

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For any fuzzy set $M \in \mathbf{L}^Y$ define

$$M^T := M \cup \bigcup \{ \quad \otimes B \mid A \sqsubseteq B \in T; S(A; M) \geq \quad \}$$

For each non-negative integer define

$$M^{T_n} := \begin{cases} M & \text{if } n = 0 \\ (M^{T_{n-1}})^T & \text{if } n \geq 1: \end{cases}$$

Define an operator $cl_T \colon \mathbf{L}^Y \to \mathbf{L}^Y$ by

$$cl_T(M) := \bigcup_{n=0}^{\infty} M^{T_n}:$$

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For each $M \in Mod(T)$ *we have* $cl_T(M) = M$.

Lemma

Let T *be a set of fAD formulas over* Y. *Further let both* Y *and* $\mathbf{L}$ *be finite. Then,* cl_T *is an* $\mathbf{L}^*$*-closure operator such that for each* $M \in \mathbf{L}^Y$, $C_{Mod(T)}(M) = cl_T(M)$.

INPUT: Set T of fAD formulas, $M \in \mathbf{L}^Y$

OUTPUT: $C_{Mod(T)}(M)$

1. if for all $A \sqsubseteq B \in T$, $(S(A;M) < \;)$ or $(S(A;M) \geq \;$ and $S(B;M) \geq \;)$, then goto step 3);
2. take any $A \sqsubseteq B \in T$ such that $S(A;M) \geq \;$ and $S(B;M) < \;$; set M to $M \cup \{\; \otimes B\}$; goto step 1);
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$$Imp(T) := \{A \Rightarrow B \mid \text{ for all } A \sqsubseteq B \in T\}:$$

$M \in \mathbf{L}^Y$ **model** of $Imp(T)$ if $||A \Rightarrow B||_M = S(A; M)^* \to S(B; M) = 1$ for each $A \Rightarrow B \in Imp(T)$.

Lemma

If we choose $= \quad = 1$, then for every set of fAD formulas T the following holds

$$Mod(Imp(T)) \subseteq Mod(T):$$

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Definition

Two sets T_1 and T_2 of fAD formulas are called **equivalent**, written $T_1 \equiv T_2$, if

$$T_1 \models {}'_2 \text{ and } T_2 \models {}'_1;$$

for each $'_1 \in T_1$ and $'_2 \in T_2$.

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Let T_1 and T_2 be sets of fAD formulas. Then, the following are equivalent:

1. $Mod(T_1) = Mod(T_2)$,
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A set T' of fAD formulas is called a **non-redundant base** of T if $T \equiv T'$ and there is no $T'' \subset T'$ with $T'' \equiv T$. A set T' of fAD formulas is called a **minimal base** of T if $T \equiv T'$ and for each T'' such that $T \equiv T''$, we have $|T'| \leq |T''|$.

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- $T' := T \setminus \{A \sqsubseteq B\}$. If $T' \models A \sqsubseteq B$, then $T \equiv T'$.

INPUT: Set $T := \{A_i \sqsubseteq B_i \mid i \in \{1, \ldots, n\}\}$ of fAD formulas
OUTPUT: T non-redundant base
for all $i := 1$ **to** n **do**
  $T' := T \setminus \{A_i \sqsubseteq B_i\}$;
  if $T \models A_i \sqsubseteq B_i$ **then** $T := T'$;
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$\{$Thank you for$\} \sqsubseteq \{$your attention!$\}$