Concept lattices of incomplete data

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Introduction

Burmeiser and Holzer (2000)

- introduce incomplete context
- use Kleene logic ($L = \{0, 1, ?\}$)
- apply to attribute implications and attribute exploration

Drawback of Kleene logic

- In Kleene logic the implication satisfies $(? \to ?) = ?$
  The implication $a \to a$ should always hold, even if the value of a is unknown.

Obiedkov (2002)

- use Modal logic instead of Kleene logic
- overcome problem of Kleene logic

Our goal

- construct a concept lattice of an incomplete context
  Requirements: contains all information and small

Outline

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Concept-Forming Operators and Complete Homomorphisms

Let

- $h\colon \mathbf{L} \to \mathbf{L}'$ be a complete homomorphism of complete residuated lattices
- $\langle X, Y, I \rangle$ be a formal $\mathbf{L}$-context

Lemma

For each $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ it holds

$$(h \circ A)^{\uparrow_{h \circ I}} = h \circ A^{\uparrow_I}, \qquad (h \circ B)^{\downarrow_{h \circ I}} = h \circ B^{\downarrow_I}.$$

Relationship of $\mathscr{B}(X,Y,I)$ and $\mathscr{B}(X,Y,h\circ I)$

Let

- $h\colon \mathbf{L}\to\mathbf{L}'$ be a complete homomorphism of complete residuated lattices
- $\langle X,Y,I\rangle$ be a formal $\mathbf{L}$-context

Theorem

- *For each formal concept $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ it holds $\langle h\circ A, h\circ B\rangle \in \mathscr{B}(X,Y,h\circ I)$.*
- *The induced mapping $h^{\mathscr{B}(X,Y,I)}\colon \mathscr{B}(X,Y,I)\to\mathscr{B}(X,Y,h\circ I)$ is a complete homomorphism.*
- *For each $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathscr{B}(X,Y,I)$ it holds*

$$\left(h^{\mathscr{B}(X,Y,I)}(\langle A_1,B_1\rangle) \preceq h^{\mathscr{B}(X,Y,I)}(\langle A_2,B_2\rangle)\right) = h(\langle A_1,B_1\rangle \preceq \langle A_2,B_2\rangle).$$

- *If h is injective, then so is $h^{\mathscr{B}(X,Y,I)}$.*
- *If h is surjective, then so is $h^{\mathscr{B}(X,Y,I)}$.*

Structure of $\mathscr{B}(X,Y,I)$

Let

- $\langle X,Y,I\rangle$ be a formal $\mathbf{L}$-context
- $\mathbf{L}$ be isomorphic to the direct product $\mathbf{L}_1 \times \mathbf{L}_2$
- $p_1 : L \to L_1$ and $p_2 : L \to L_2$ be the respective projections

Theorem

- *$\mathscr{B}(X,Y,I)$ is isomorphic to the direct product $\mathscr{B}(X,Y,p_1 \circ I) \times \mathscr{B}(X,Y,p_2 \circ I)$.*
- *The mappings $p_1^{\mathscr{B}(X,Y,I)}$ and $p_2^{\mathscr{B}(X,Y,I)}$ correspond to the respective Cartesian projections.*

Outline

Boolean Algebras

Example

Two-element Boolean algebra $\mathbf{2}$

Example

Boolean algebra of all mappings from a set U to $\mathbf{2}$, denoted by $\mathbf{2}^U$

Boolean Algebras as Residuated Lattices

Let $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1 \rangle$ be a Boolean algebra

We set:

$$a \otimes b = a \wedge b, \qquad a \to b = a' \vee b,$$

$\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a residuated lattice

Boolean Algebras with Variables and Assignments

Let $\mathbf{L}$ be a finite Boolean algebra
together with a set of *variables* $U = \{u_1, \ldots, u_k\}$
and mapping $\iota : U \to L$, such that $\mathbf{L}$ is generated by $\iota(U)$.

Definition (Boolean algebras with variables)

- We call $\mathbf{L}$ a *Boolean algebra with variables* $u_1, \ldots, u_k$.
- If $\mathbf{L}$ is freely generated by $\iota(U)$, then the variables $u_1, \ldots, u_k$ are said to be *independent*.

Definition (assignments)

Mappings $v : U \to \mathbf{2}$ are called *assignments*.

Since $\mathbf{L}$ is generated by $\iota(U)$, then for each assignment v there exists at most one homomorphism $\bar{v} : \mathbf{L} \to \mathbf{2}$, such that $v = \bar{v} \circ \iota$.

Definition (admissible assignments)

If this homomorphism exists, then the assignment v is called *admissible*.

Admissible Assignments

- The variables $u_1, \ldots, u_k$ are independent if and only if each assignment is admissible.
- The other cases: dependencies between the unknown values

Example

If $\imath(u_1) \leq \imath(u_2)$, then there is no admissible assignment v such that $v(u_1) = 1$ and $v(u_2) = 0$.

Example

Let $\mathbf{L}$ be a Boolean algebra with variables u_1, u_2,
where assignment v with $v(u_1) = v(u_2) = 1$ is admissible.
Further, let $X = \{x_1, x_2, x_3\}$ be a set and $A = \{x_1, {}^{u_1 \vee u_2'}/x_2, {}^{u_1'}/x_3\}$ be an $\mathbf{L}$-set in X.

Then we obtain $\bar{v} \circ A = \{x_1, x_2\}$.

Construction of Boolean Algebras with Variables

Two natural requirements

1. Generality: for every possible dependency between variables there exists an appropriate Boolean algebra $\mathbf{L}$ (together with a mapping $\iota : U \to L$).
2. Efficiency: the Boolean algebra $\mathbf{L}$ is (up to isomorphism) the smallest element within the class of residuated lattices satisfying the generality requirement 1.

Theorem

The following holds for each subset $V \subseteq \mathbf{2}^U$.
1. Let $\mathbf{L} = \mathbf{2}^V$. Then there is a mapping $\iota : U \to L$ satisfying: For each $v \in V$ there is exactly one homomorphism $\bar{v} : \mathbf{L} \to \mathbf{2}$ such that $v = \bar{v} \circ \iota$.
2. Let $\mathbf{L}'$ be a residuated lattice and $\iota' : U \to L'$ a mapping such that for each $v \in V$ there is a homomorphism $\bar{v}' : \mathbf{L}' \to \mathbf{2}$ satisfying $v = \bar{v}' \circ \iota'$. Then there exists a surjective homomorphism of residuated lattices $s : \mathbf{L}' \to \mathbf{L}$ such that for each $v \in V$ it holds $\bar{v}' = \bar{v} \circ s$.

Example of Boolean Algebras with Variables

Let

- $U = \{u_1, u_2\}$
- $V = \{v_1, v_2, v_3\}$ with
  $v_1(u_1) = v_1(u_2) = 0$, $v_2(u_1) = 0$, $v_2(u_2) = 1$, and $v_3(u_1) = v_3(u_2) = 1$

$\mathbf{L} = \mathbf{2}^V$ is a Boolean algebra with variables u_1, u_2, where $\iota(u_1) \leq \iota(u_2)$

	u_1	u_2	0	$u_1' \wedge u_2$	u_2'	$u_1 \vee u_2'$	u_1'	1
$\bar{v}_1$	0	0	0	0	1	1	1	1
$\bar{v}_2$	0	1	0	1	0	0	1	1
$\bar{v}_3$	1	1	0	0	0	1	0	1

Elements of $\mathbf{L}$ are written in columns, their values in the mappings $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ are in rows.

Outline

Incomplete Contexts

Let $\mathbf{L}$ be a Boolean algebra with variables $u_1, \ldots, u_k$, $U = \{u_1, \ldots, u_k\}$

Suppose: $U \subseteq L$, $\imath$ is the inclusion $U \to L$.

Definition (incomplete context)

An *incomplete* $\mathbf{L}$*-context* is a triple $\langle X, Y, I \rangle$, where X and Y are sets and $I\colon X \times Y \to L$ is an $\mathbf{L}$-relation such that $I(X \times Y) \subseteq U \cup \{0, 1\}$.

Example

Boolean algebra $\mathbf{L}$ with variables u_1, u_2 is from the example.

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2	u_1	×	u_2	×	
x_3	×	×	×		
x_4		×			

Completions

Definition (completion)

An ordinary formal context $\langle X, Y, J\rangle$ is a *completion* of $\langle X, Y, I\rangle$, if $J = \bar{v} \circ I$ for an admissible assignment $v : U \to \mathbf{2}$.

Example

$v_1(u_1) = v_1(u_2) = 0$

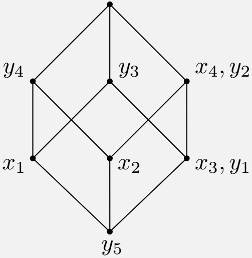
$\langle X, Y, I\rangle$

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2	u_1	×	u_2	×	
x_3	×	×	×		
x_4		×			

$\langle X, Y, \bar{v}_1 \circ I\rangle$

	y_1	y_2	y_3	y_4	y_5
x_1			×	×	
x_2		×		×	
x_3	×	×	×		
x_4		×			

$\mathscr{B}(X, Y, \bar{v}_1 \circ I)$



Outline

$\mathbf{L}$-concept Lattice of an Incomplete $\mathbf{L}$-context

Let $\langle X, Y, I\rangle$ be incomplete $\mathbf{L}$-context.
$\mathscr{B}(X,Y,I)$ is the $\mathbf{L}$-concept lattice of $\mathbf{L}$-context $\langle X, Y, I\rangle$
Let v be an admissible assignment. $\bar{v}^{\mathscr{B}(X,Y,I)}$ is surjective.

Each $\langle A_0, B_0\rangle \in \mathscr{B}(X,Y,\bar{v}\circ I)$ can be obtained as the image of a concept $\langle A, B\rangle \in \mathscr{B}(X,Y,I)$ in the mapping $\bar{v}^{\mathscr{B}(X,Y,I)}$.

$\mathbf{L}$ is isomorphic to $\mathbf{2}^V$

Theorem

$\mathscr{B}(X,Y,I)$ is isomorphic to the direct product $\prod_{v\in V}\mathscr{B}(X,Y,\bar{v}\circ I)$. The mappings $\bar{v}^{\mathscr{B}(X,Y,I)}: \mathscr{B}(X,Y,I)\to\mathscr{B}(X,Y,\bar{v}\circ I)$ correspond to the respective Cartesian projections.

Example

The $\mathbf{L}$-concept lattice $\mathscr{B}(X,Y,I)$ of the incomplete context from the example is isomorphic to the direct product $\mathscr{B}(X,Y,\bar{v}_1\circ I)\times\mathscr{B}(X,Y,\bar{v}_2\circ I)\times\mathscr{B}(X,Y,\bar{v}_3\circ I)$ and has $8\cdot 8\cdot 7 = 448$ elements.

Crisply Generated Concepts of an Incomplete $\mathbf{L}$-context

Definition (Bělohlávek, Sklenář, Zacpal)

A formal $\mathbf{L}$-concept $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ is called *crisply generated*, if there is a crisp set $B_0 \subseteq Y$ such that $A = (B_0)^{\downarrow}$ (and hence $B = (B_0)^{\downarrow\uparrow}$). The set of all crisply generated $\mathbf{L}$-concepts is denoted by $\mathscr{B}_c(X,Y,I)$.

$\mathscr{B}_c(X,Y,I)$ is a complete lattice

Theorem

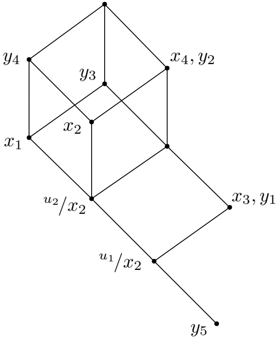
For each $v \in V$, the restriction $\bar{v}^{\mathscr{B}_c(X,Y,I)} : \mathscr{B}_c(X,Y,I) \to \mathscr{B}(X,Y,v \circ I)$ of $\bar{v}^{\mathscr{B}(X,Y,I)}$ is

- *a surjective,*
- *$\bigwedge$-preserving mapping, such that*
- *for each $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathscr{B}_c(X,Y,I)$ it holds $\bar{v}^{\mathscr{B}_c(X,Y,I)}(\langle A_1,B_1\rangle) \leq \bar{v}^{\mathscr{B}_c(X,Y,I)}(\langle A_2,B_2\rangle)$ iff $\bar{v}(\langle A_1,B_1\rangle \preceq \langle A_2,B_2\rangle) = 1$.*

Example of $\mathscr{B}_c(X,Y,I)$

$\langle X,Y,I\rangle$	$\mathscr{B}_c(X,Y,I)$

	y_1	y_2	y_3	y_4	y_5
x_1			$\times$	$\times$	
x_2	u_1	$\times$	u_2	$\times$	
x_3	$\times$	$\times$	$\times$		
x_4		$\times$			



Elements of $\mathscr{B}_c(X,Y,I)$:

$$\langle\{x_1,x_2,x_3,x_4\},\emptyset\rangle, \langle\{x_1,x_2\},\{{}^{u_2}/y_3,y_4\}\rangle, \langle\{x_1,{}^{u_2}/x_2,x_3\},\{y_3\}\rangle,$$

$$\langle\{x_2,x_3,x_4\},\{y_2\}\rangle, \langle\{x_1,{}^{u_2}/x_2\},\{y_3,y_4\}\rangle, \langle\{x_2\},\{{}^{u_1}/y_1,y_2,{}^{u_2}/y_3,y_4\}\rangle,$$

$$\langle\{{}^{u_2}/x_2,x_3\},\{{}^{u_1\vee u_2'}/y_1,y_2,y_3\}\rangle, \langle\{{}^{u_2}/x_2\},\{{}^{u_1\vee u_2'}/y_1,y_2,y_3,y_4,{}^{u_2'}/y_5\}\rangle,$$

$$\langle\{{}^{u_1}/x_2,x_3\},\{y_1,y_2,y_3\}\rangle, \langle\{{}^{u_1}/x_2\},\{y_1,y_2,y_3,y_4,{}^{u_1'}/y_5\}\rangle, \langle\emptyset,\{y_1,y_2,y_3,y_4,y_5\}\rangle$$

Size of $\mathscr{B}_c(X,Y,I)$

- in the worst case - exponentially
- experiments - linear (for small number of variables)

Results of experiments

- The Digits Context (48 concepts)

No. of variables	1	2	3	4	5	6	7	8	9	10
Avg. no. of concepts	50.31	54.31	56.26	59.38	63.49	65.84	68.21	71.01	73.05	77.39
Std. deviation	3.16	6.34	6.51	8.00	8.19	8.67	10.37	11.71	11.31	11.67

- The Tea Ladies Context (65 concepts)

No. of variables	1	2	3	4	5	6	7
Avg. no. of concepts	67.88	71.54	74.73	78.3	82.12	86.09	89.67
Std. deviation	2.56	3.79	4.66	5.0	7.53	7.02	8.44

Outline

Conclusions and Future Work

Conclusions

- Our approach could find practical applications.

Future work

- Attribute implications in our framework.
- More results on the size of concept lattices of incomplete contexts.

Remark

- Generalizing the theory to FCA in fuzzy setting was done. Article was sent to IPMU 2012.