What is a negation of a fuzzy attribute?

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Outline

- Introduction
- Reducibility in classical case
- Failure of reducibility in fuzzy case
- Common view: reducibility fails due to lack of double negation
- New view: The problem is with the notion of negation of a fuzzy attribute, not with the law of double negation
- New notion of negation of fuzzy attribute and what can be saved

Introduction

As is well-known, a given ordinary binary relation $I \in \{0,1\}^{X\times Y}$ (representing e.g. a yes/no relationship between objects $x \in X$ and attributes $y \in Y$) induces two important pairs of operators between $\{0,1\}^X$ and $\{0,1\}^Y$.

Namely, a pair $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$ defined by

$$\begin{aligned} A^{\uparrow_I} &= \{y \in Y \mid \text{ for each } x \in A : \langle x,y\rangle \in I\}, \\ B^{\downarrow_I} &= \{x \in X \mid \text{ for each } y \in B : \langle x,y\rangle \in I\} \end{aligned} \quad (1)$$

and a pair $\langle {}^{\cap_I}, {}^{\cup_I} \rangle$ defined by

$$\begin{aligned} A^{\cap_I} &= \{y \in Y \mid \text{ there exists } x \in A \text{ such that } \langle x,y\rangle \in I\}, \\ B^{\cup_I} &= \{x \in X \mid \text{ for each } y \in Y : \langle x,y\rangle \in I \text{ implies } y \in B\} \end{aligned} \quad (2)$$

It is well known that the two pairs of operators are mutually definable (e.g. Duntsch and Gediga, ICDM 2002).

An important consequence:

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \text{ and } \mathcal{B}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I) \text{ are isomorphic as lattices} \tag{3}$$

(equivalently, $\mathcal{B}(X^{\cap_{\neg I}}, Y^{\cup_{\neg I}}, \neg I)$ and $\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ are isomorphic), with $\langle A, B\rangle \mapsto \langle A, \neg B\rangle$ being an isomorphism.

Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I). \tag{4}$$

However, as shown by Georgescu and Popescu (AML 2004), when fuzzy relations instead of ordinary relations I are considered, the above mutual reducibility fails to hold.

That is:

With complete residuated lattices as structures of truth degrees, i.e. $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that ...

A fuzzy relation $I \in L^{X\times Y}$ induces two pairs of operators between L^X and L^Y, i.e. the sets of all fuzzy sets in X and Y, defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)), \quad B^{\downarrow_I}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)),$$
$$A^{\cap_I}(y) = \bigvee_{x\in X}(A(x) \otimes I(x,y)), \quad B^{\cup_I}(x) = \bigwedge_{y\in Y}(I(x,y) \to B(y)),$$

Then **it is not true that**

$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices.

Here, the complement $\neg I$ of a fuzzy relation I is defined by

$$\neg I(x,y) = I(x,y) \to 0.$$

That is, one uses the CLASSICAL truth function $\neg$ of negation defined by

$$\neg a = a \to 0.$$

New notion of complement

The classical notion of a complement $\neg I$ of a fuzzy relation may be looked at the following way:
Each attribute $y \in Y$ in the data table representing I is replaced by its complement. That it, each fuzzy set $I_y \in L^X$, representing attribute y, defined by $I_y(x) = I(x, y)$ is replaced in the table by its complement $\neg I_y$ defined by

$$(\neg I_y)(x) = \neg(I_y(x)), \qquad \text{i.e. } (\neg I_y)(x) = I_y(x) \to 0.$$

The complement is in fact the residuum of a w.r.t. 0. However, one may also consider a residuum of $a \in L$ w.r.t. to an arbitrary element $b \in L$, i.e. one may consider

$$\neg_b a = a \to b, \tag{5}$$

of which $\neg a$ is a particular case because $\neg a = \neg_0 a$. In addition to $\neg I_y$, the "negation relative to 0" one may therefore also consider $\neg_b I_y$, the "negation relative to b", for other degrees b, defined by

$$(\neg_b I_y)(x) = \neg_b(I_y(x)), \qquad \text{i.e. } (\neg_b I_y)(x) = I_y(x) \to b.$$

For every original attribute y, I_y may therefore be replaced not just by the complement $\neg_0 I_y$ w.r.t. 0 but by several complements $\neg_b I_y$ w.r.t. $b \in K$ with $K \subseteq L$ being a set of selected values, bringing us the following definition.

Definition

For a set $K \subseteq L$, the K-complement of a fuzzy relation I between X and Y is a fuzzy relation $\neg_K I$ between X and $Y \times K$ defined by

$$(\neg_K I)(x, \langle y, b\rangle) = \neg_b I(x, y)$$

for every $x \in X$, $y \in Y$, and $b \in K$.

Ordinary complement is just $\{0\}$-complement.

With this notion of complement, the above mentioned reducibility holds:

Theorem

For a fuzzy relation I between X and Y, let $\neg I$ denote $\neg_{L-\{1\}} I$. Then $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y \times (L-\{1\})^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices, with the mappings $\langle A, B\rangle \mapsto \langle A, D\rangle$, where

$$D(y,b) = \neg_b B(y)$$

for $y \in Y$, $b \in L-\{1\}$, and $\langle A, D\rangle \mapsto \langle A, B\rangle$, where

$$B(y) = \bigwedge_{b \in L-\{1\}} \neg_b D(y,b)$$

for $y \in Y$, being the isomorphism and its inverse. Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L-\{1\})^{\downarrow_{\neg I}}, \neg I).$$

For $L = \{0,1\}$, the theorem yields the classical reducibility theorem.

Remarks

(a) One easily checks that since $\neg_1 I_y(x) = 1$ for each $x \in X$, one may replace $L - \{1\}$ by L in Theorem.

(b) A converse statement to Theorem does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is equal to $\mathrm{Ext}(X^{\cap_{\sim I}}, Z^{\cup_{\sim I}}, {\sim}I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators $^{\cap_J}$ and $^{\cup_J}$ (Belohlavek, Konecny, AI 2011).

(c) Theorem generalizes classical reducibility theorem but its proof does not use the law of double negation.

Future Work

- Study role of the presented negation of fuzzy attributes in FCA.
- Study role of the presented negation of fuzzy attributes in fuzzy logic.

Relevant references

- Belohlavek, Konecny: Concept lattices of isotone vs. antitone Galois connections in graded setting: mutual reducibility revisited (submitted).
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