Factorization of lattices by non-compatible tolerances

Michal Krupka (Palacký University)

DATA ANALYSIS AND MODELING LAB

Palacky University, Olomouc, Czech Republic

european social fund in the czech republic

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Introduction

- We study the problem of reducing the size of concept lattice by "adding crosses" to incidence relation.
- Special case: aggregation of attributes.
- Classical theory (complete tolerances and block relations) is too limiting.

Outline

Complete tolerances

Definition (tolerance, blocks, factor set)

- *A tolerance on* X is a relation $\sim$ which is reflexive and symmetric.
- *A block of* $\sim$ is a set $B \subseteq X$ such that if $a, b \in B$, then $a \sim b$.
- *Factor set* $X/\sim$ is the set of all maximal blocks of $\sim$

Definition

A *complete tolerance* on a complete lattice $\mathbf{L}$: from $a_j \sim b_j$ for all $j \in J$ it follows $\bigvee_{j\in J} a_j \sim \bigvee_{j\in J} b_j$ and $\bigwedge_{j\in J} a_j \sim \bigwedge_{j\in J} b_j$.

Theorem (Czédli, Wille)

$\mathbf{L}/\sim$ *with the ordering*

$$B_1 \leq B_2 \quad \textit{iff} \quad \bigvee B_1 \leq \bigvee B_2 \quad (\textit{iff} \quad \bigwedge B_1 \leq \bigwedge B_2)$$

is a complete lattice.

Block relations

Let $\langle X, Y, I \rangle$ be a (classical) formal context.

Definition

A relation $J \supseteq I$ is a *block relation*, if for any $x \in X$ and $y \in Y$:

- $\{x\}^{\uparrow_J}$ is an intent of $\langle X, Y, I \rangle$,
- $\{y\}^{\downarrow_J}$ is an extent of $\langle X, Y, I \rangle$.

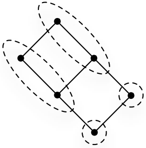
Theorem (Wille)

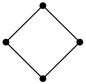
1. *The lattice of all block relations on* $\langle X, Y, I \rangle$ *is isomorphic to the lattice of all complete tolerances on* $\mathcal{B}(X, Y, I)$. *For a block relation* J, *the corresponding tolerance* $\sim^J$ *is given by* $\langle A_1, B_1 \rangle \sim^J \langle A_2, B_2 \rangle$ *iff* $(A_1 \cup A_2) \times (B_1 \cup B_2) \subseteq J$.
2. *The lattices* $\mathcal{B}(X, Y, J)$ *and* $\mathcal{B}(X, Y, I)/{\sim^J}$ *are isomorphic.*

Block relations: examples

I and J — $\mathcal{B}(X,Y,I)$ — $\mathcal{B}(X,Y,J)$

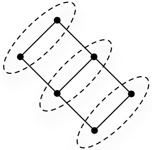
		*	
	x	*	
	x	x	
		x	x



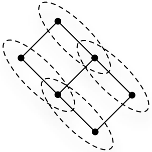


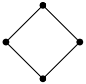
I and J — $\mathcal{B}(X,Y,I)$ — $\mathcal{B}(X,Y,J)$

	*		
	x		
	x	x	
*	*	x	x

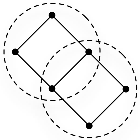


		*	
	x	*	
*	x	x	*
		x	x





	*	*	
	x	*	
*	x	x	*
*	*	x	x



Notes

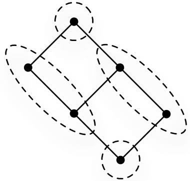
- I is depicted by crosses (x), J is depicted by crosses and asterisks (x, $*$)
- dashed ovals denote maximal blocks of $\sim^J$
- $\mathcal{B}(X,Y,J)$ is isomorphic with $\mathcal{B}(X,Y,I)/{\sim^J}$

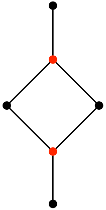
Outline

Example

I and J | $\mathcal{B}(X,Y,I)$ | $\mathcal{B}(X,Y,J)$

	x	*	
	x	x	*
		x	x





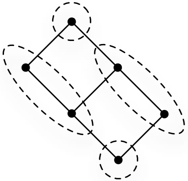
Some problems:

- Which tolerances on $\mathcal{B}(X,Y,I)$ can be described by means of relations $J \supseteq I$?
- What is the connection between $\mathcal{B}(X,Y,I)/{\sim^J}$ and $\mathcal{B}(X,Y,J)$?
- The problem of superfluous concepts in $\mathcal{B}(X,Y,J)$.

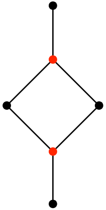
Example

I and J			
	x	$*$	$*$
	x	x	$*$
		x	x

$\mathcal{B}(X, Y, I)$



$\mathcal{B}(X, Y, J)$



Some problems:

- Which tolerances on $\mathcal{B}(X, Y, I)$ can be described by means of relations $J \supseteq I$?
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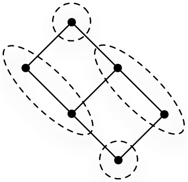
- The problem of superfluous concepts in $\mathcal{B}(X, Y, J)$.

Example

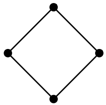
I and J

	X	*	*
	X	X	*
		X	X

$\mathcal{B}(X, Y, I)$



$\mathcal{B}(X, Y, J)$



Some problems:

- Which tolerances on $\mathcal{B}(X, Y, I)$ can be described by means of relations $J \supseteq I$?
- What is the connection between $\mathcal{B}(X, Y, I)/{\sim^J}$ and $\mathcal{B}(X, Y, J)$?
- The problem of superfluous concepts in $\mathcal{B}(X, Y, J)$.

Outline

Posets with tolerance

Definition (Poset with tolerance, poset with equivalence)

A *poset with tolerance* is a structure $\mathbf{U} = \langle U, \leq, \sim \rangle$ where $\langle U, \leq \rangle$ is a partially ordered set and $\sim$ is a tolerance on U. If $\sim$ is an equivalence then $\mathbf{U}$ is a *poset with equivalence*.

Relation $\leq_{\sim}^{n}$ on $\mathbf{U}$

- $u_1 \leq_{\sim}^{0} u_2$ iff $u_1 \leq u_2$
- for any integer $n > 0$, $u_1 \leq_{\sim}^{n} u_2$ iff there exist $v_1, v_2 \in U$ such that $u_1 \leq_{\mathbf{U}}^{n-1} v_1$, $v_1 \sim v_2$, and $v_2 \leq u_2$
- $u_1 \leq_{\sim}^{\infty} u_2$

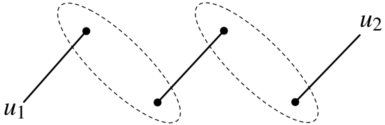


Figure: $u_1 \leq_{\sim}^{2} u_2$

Diagonal property

Definition (Diagonal property)

From $v_1 \lesssim^1 v_2$ and $v_2 \lesssim^1 v_1$ it follows $v_1 \sim v_2$.

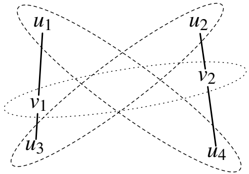


Figure: Diagonal property: if elements in dashed ovals are related, then those in the dotted one are also related.

Definition (Strong diagonal property)

For each integer $n > 1$ from $v_1 \lesssim^n v_2$ and $v_2 \lesssim^n v_1$ it follows $v_1 \sim v_2$.

Observation: $\sim$ is an equivalence.

Completions of posets with tolerance

Let $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$.

Definition (k-complete poset with tolerance)

1. $\langle U, \leq \rangle$ is a complete lattice,
2. $\sim$ is a complete tolerance on $\langle U, \leq \rangle$,
3. $\sim^k = U \times U$ (where $\sim^\infty = U \times U$ by definition).

Definition (k-embedding $f\colon \mathbf{U} \to \mathbf{V}$)

for each integer $n < k$:

$$u_1 \leq^n_{\sim_\mathbf{U}} u_2 \quad \text{iff} \quad f(u_1) \leq^n_{\sim_\mathbf{V}} f(u_2).$$

Definition (k-completion of a poset with tolerance $\mathbf{U}$)

is a k-embedding $f\colon \mathbf{U} \to \mathbf{V}$ where $\mathbf{V}$ is a k-complete poset with tolerance. The k-completion $f\colon \mathbf{U} \to \mathbf{V}$ is called *minimal*, if for any other k-completion $f'\colon \mathbf{U} \to \mathbf{V}'$ there exists a k-embedding $h\colon \mathbf{V} \to \mathbf{V}'$ such that $f' = h \circ f$.

Basic residuated lattices

We use the following three structures:

(1) $(n+1)$**-element Łukasiewicz chain**

$L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ with Łukasiewicz operations: $a \otimes b = \max(a + b - 1, 0)$,

(2) Three-element Gödel chain

$L = \{0, 0.5, 1\}$ with Gödel operations: $a \otimes b = \min(a, b)$,

(3) Countable Goguen chain

$L = \{2^{-n} \mid n = 0, 1, 2, \ldots\} \cup \{0\}$ with Goguen operations: $a \otimes b = ab$.

The second greatest element $g \in L$

- $g = \frac{n-1}{n}$ in (1), $g = 0.5$ in (2), (3)
- $g^2 < g$ in (1), (3), $g^2 = g$ in (2)
- $g^{n-1} > 0$ and $g^n = 0$ in (1), $g^k > 0$ for each k in (2), (3)
- we write $g^\infty = 0$

Completely lattice $\mathbf{L}$-ordered sets

Definition ($\mathbf{L}$-ordered set)

is a tuple $\langle\langle V, \approx\rangle, \preceq\rangle$, where

- $\approx$ is an $\mathbf{L}$-equality on V,
- $\preceq$ which is compatible with $\approx$, reflexive, transitive,
- and $(v \preceq w) \wedge (w \preceq v) \leq v \approx w$ (*antisymmetry*).

Definition (Embedding of $\mathbf{L}$-ordered sets $\mathbf{U}$ and $\mathbf{V}$)

is a mapping $f\colon U \to V$ such that for each $u_1, u_2 \in U$,

$$u_1 \preceq u_2 = f(u_1) \preceq f(u_2).$$

Definition (Completely lattice $\mathbf{L}$-ordered set)

Each $\mathbf{L}$-set in V has a supremum and infimum.

We have Dedekind-MacNeille completion of $\mathbf{L}$-ordered sets.

$\mathbf{L}$-extensions of posets with tolerance

Definition ($\mathbf{L}$-extension of $\mathbf{U}$)

is an $\mathbf{L}$-ordered set $\langle\langle U, \approx\rangle, \preceq\rangle$ such that ${}^1\preceq = \leq$ and ${}^g\approx = \sim$. The $\mathbf{L}$-extension is *minimal* if $\preceq$ is minimal (w.r.t. $\mathbf{L}$-set inclusion).

Observation: If $g^2 = g$ then $\sim$ should be an equivalence.

Set $\mathbf{U}_\mathbf{L} = \langle\langle U, \approx\rangle, \preceq\rangle$, where for each $u_1, u_2 \in U$,

- $u_1 \preceq u_2 = g^n$, where n is the least element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$ such that $u_1 \leq^n_{\mathbf{U}} u_2$,
- $u_1 \approx u_2 = (u_1 \preceq u_2) \wedge (u_2 \preceq u_1)$.

Theorem

$\mathbf{U}_\mathbf{L}$ *is an* $\mathbf{L}$*-ordered set. Moreover, the following conditions are equivalent:*

(a) $\mathbf{U}$ *has an* $\mathbf{L}$*-extension.*

(b) $\mathbf{U}$ *satisfies the diagonal property and if* $g^2 = g$*, then* $\mathbf{U}$ *satisfies the strong diagonal property.*

(c) $\mathbf{U}_\mathbf{L}$ *is the minimal* $\mathbf{L}$*-extension of* $\mathbf{U}$*.*

Main results on completions of posets with tolerance

Let k be the least element of $\{1, 2, \ldots\} \cup \{\infty\}$ such that $g^k = 0$ (for $g^2 = g$ we set $k = \infty$).

Theorem

Let $\mathbf{U_L}$, $\mathbf{V_L}$ be minimal $\mathbf{L}$-extensions. Then $f : U \to V$ is a k-embedding of $\mathbf{U}$ into $\mathbf{V}$, if and only if it is an embedding of the $\mathbf{L}$-ordered set $\mathbf{U_L}$ into $\mathbf{V_L}$.

Theorem

Suppose that if $g^2 = g$ then $\sim$ is an equivalence. Then $\mathbf{U}$ is k-complete, if and only if it has the minimal $\mathbf{L}$-extension and this minimal $\mathbf{L}$-extension is a completely lattice $\mathbf{L}$-ordered set.

Theorem

1. Any poset with tolerance satisfying the diagonal property has the minimal k-completion for any $k \in \{1, 2, \ldots\} \cup \{\infty\}$.
2. Any poset with equivalence satisfying the strong diagonal property has the minimal completion.

Outline

Translating the results to formal contexts

$\mathcal{B}(X, Y, I)$: (classical) concept lattice, $\sim$ a tolerance on $\mathcal{B}(X, Y, I)$, induced by $J \supseteq I$.

Theorem

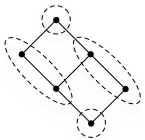
$\sim$ *satisfies the diagonal property.*

Let $I^J_{\mathbf{L}} \colon X \times Y \to L$ be an $\mathbf{L}$-relation such that $I^J_{\mathbf{L}}(x, y) = g^n$, where $n \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ is the least element satisfying

$$\langle \{x\}^{\uparrow_I \downarrow_I}, \{x\}^{\uparrow_I} \rangle \lesssim^n \langle \{y\}^{\downarrow_I}, \{y\}^{\downarrow_I \uparrow_I} \rangle.$$

original I and J | $\mathcal{B}(X, Y, I)$ | new $\mathbf{L}$-relation $I^J_{\mathbf{L}}$

	×	∗	
	×	×	∗
		×	×



0	0	0	0
0	1	g	g^2
0	1	1	g
0	0	1	1

Problem: how to efficiently compute $I^J_{\mathbf{L}}$? (However, in 3-element Łukasiewicz chain $g^2 = 0$.

Main results

Let $\mathbf{U}$ be the concept lattice $\mathcal{B}(X, Y, I)$ together with the tolerance $\sim$, $\mathbf{V}$ be the crisp part of $\mathcal{B}(X, Y, I_{\mathbf{L}}^{J})$ together with ${}^{g}\!\approx$.

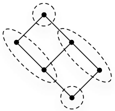
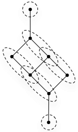
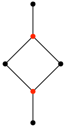
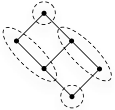
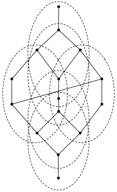
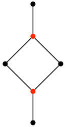
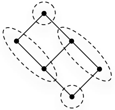
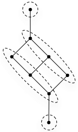
Theorem (outline of main results)

- *If $\mathbf{L}$ is the countable Goguen chain or the three-element Łukasiewicz chain then $\mathbf{V}$ is a k-completion of $\mathbf{U}$.*
- *If $\mathbf{L}$ is the countable Goguen chain then $\mathbf{V}$ is the minimal completion of $\mathbf{U}$.*
- *If $\mathbf{L}$ is the three-element Łukasiewicz chain then $\mathbf{V}/\!\sim$ is isomorphic to $\mathcal{B}(X, Y, J)$.*
- *If $\mathbf{L}$ is the three-element Gödel chain and $\sim^{J}$ is an equivalence satisfying the strong diagonal property then $\mathbf{V}$ is a completion and there are no superfluous concepts in $\mathcal{B}(X, Y, J)$.*

Main results

- Factorization of concept lattices by incomplete tolerances can be performed in two steps:
  1. computing a completion of the lattice (i.e. embedding the lattice into some complete lattice with complete tolerance),
  2. factorizing the resulting lattice.
- thus, the factor set is a subset of a complete lattice,
- the completion depends on the choice of $\mathbf{L}$,
- in some cases, there are no superfluous elements (further research needed).

Example

	original lattice	new **L**-relation $I_{\mathbf{L}}^{J}$	$\mathcal{B}(X, Y, I_{\mathbf{L}}^{J})$	factorization
Goguen		see below		
Łukasiewicz		see below		
Gödel		see below		

Goguen, new **L**-relation $I_{\mathbf{L}}^{J}$:

0	0	0	0
0	1	0.5	0.25
0	1	1	0.5
0	0	1	1

Łukasiewicz, new **L**-relation $I_{\mathbf{L}}^{J}$:

0	0	0	0
0	1	0.5	0
0	1	1	0.5
0	0	1	1

Gödel, new **L**-relation $I_{\mathbf{L}}^{J}$:

0	0	0	0
0	1	0.5	0.5
0	1	1	0.5
0	0	1	1

Outline

Conclusions and future work

Conclusions

- we presented theoretical results for factorization of concept lattices by incompatible (incomplete) tolerances
- FCA in fuzzy setting can be useful even in the classical case

Future work

- algorithms for computing $I^J_{\mathbf{L}}$
- a more comprehensive theory of completions of posets with tolerance; explain the role of fuzzy logic (a paper is in preparation)
- the problem of superfluous concepts (it seems that it can be solved)
- generalization to FCA in fuzzy setting (some results already exist)