Some applications of uncertainty measures in image processing

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Definition of sign-based image representation

A image is a integer-valued function $f(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$, defined in points $\Omega = I_N \times I_M = \{1, \ldots, N\} \times \{1, \ldots, M\}$. Notation: $\mathcal{F}$ is the set of all images of the type $f : \Omega \to \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

Definition

A relation $\tau \subseteq \Omega \times \Omega$ is called an sing-based image representation $f \in \mathcal{F}$ if the following conditions hold:

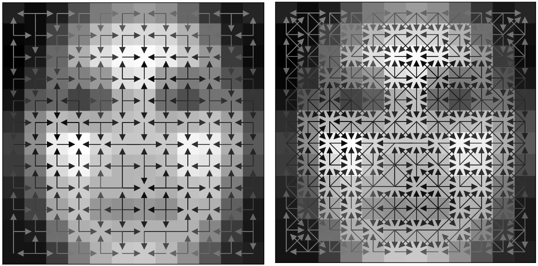
1. if $(\boldsymbol{x}, \boldsymbol{y}) \in \tau$, then $f(\boldsymbol{x}) \leqslant f(\boldsymbol{y})$;
2. if $(\boldsymbol{x}, \boldsymbol{y}) \in \tau$, $(\boldsymbol{y}, \boldsymbol{x}) \notin \tau$, then $f(\boldsymbol{x}) < f(\boldsymbol{y})$.

Notation: $\mathcal{F}_\tau$ is the set of all images with sign representation τ. Clearly, $\mathcal{F}_{\tau^{Tr}} = \mathcal{F}_\tau$,where τ^{Tr} is a transitive closure of τ. This allows us to consider next only quasi-orders, i.e. reflexive transitive relations.

Examples of sign-based representations

- A sign-based representation is called *complete*, if it is connected, i.e. any pair of points from Ω is comparable.
- If the relation τ contains only pairs of points located on a distance that is equal or lower than a threshold ε, then τ is a *neigborhood* sign-based representation:

$$\tau = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \Omega^2 \mid f(\boldsymbol{x}) \leqslant f(\boldsymbol{y}), \|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \varepsilon \right\}.$$



Axioms for image information functional

Axiom 1

An information measure is a functional $U\colon \mathcal{F} \to [0, +\infty)$.

Axiom 2

Let $f\colon \Omega \to \mathbb{Z}_+$ and let the set of values $F(\Omega) = \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega\}$ of a function f is a singleton, i.e. $|F(\Omega)| = 1$. Then $U(f) = 0$.

Axiom 3

Let $f\colon \Omega_1 \to \mathbb{Z}_+$ and $\psi\colon \Omega_1 \to \Omega_2$ be a bijection. Then $U(\psi \circ f) = U(f)$.

Acciording to Axiom 3 the mixing of pixels in an image does not change its informativity.

Axioms for image information functional. Continuation

Axiom 4

Let $f\colon \Omega_1 \to \mathbb{Z}_+$ and let $\varphi\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ be a bijection. Then $U\left(f \circ \varphi\right) = U(f)$.

By Axiom 4 the image transformation, linked with an assignment of new brightness values to pixels by bijective mappings, does not change its informativity.

For each image f let us introduce into consideration the brightness histogram $h_f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$, where $h_f(i)$ - is the pixels number with brightness i.

Corollary 1

Given images $f : \Omega_1 \to \mathbb{Z}_+$ и $g : \Omega_2 \to \mathbb{Z}_+$. Then $U(f) = U(g)$, if there is a bijection $\varphi\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $h_g(i) = h_f\left(\varphi(i)\right)$ for any $i \in \mathbb{Z}_+$.

Axioms for image information functional. Continuation

Let us introduce functional $\bar{U}(f) = \frac{U(f)}{|\Omega|}$ showing the mean value of pixel informativity in f.
If an image g consists of k copies of an image f, then the frequency $p_g(i)$ of pixels appearence with brightness i in g is equal to the frequency $p_f(i)$ of pixels appearence with brightness i in the image f .
We can assume that $\bar{U}(f) = \bar{U}(g)$ for such images. This leads to the following axiom.

Axiom 5

Let $f, g \in \mathcal{F}$ и $p_g(i) = p_f(i)$ for all $i \in \mathbb{Z}_+$. Then $\bar{U}(f) = \bar{U}(g)$.

Axioms for image information functional. Continuation

Let $\varphi\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ be an arbitrary mapping, and let $f \circ \varphi$ be an image of f under mapping φ. If φ is not an injection, then we lose the part of information about brightness levels in the initial image f. Such trasformations are essential in image processing in a problem of decreasing brightness levels, that are usually used for segmentation. Let $\varphi\left(f\left(\Omega\right)\right) = \{b_1, \ldots, b_n\}$ and $\Omega_k = \{\boldsymbol{x} \in \Omega \mid \varphi\left(f\left(\boldsymbol{x}\right)\right) = b_k\}$. Next axiom assumes that uncertainty is increased additively under such transformations.

Axiom 6

Let $f\colon \Omega \to \mathbb{Z}_+$ and $\varphi\colon \mathbb{Z}_+ \to \mathbb{Z}_+$. Assume also that $\varphi\left(f\left(\Omega\right)\right) = \{b_1, \ldots, b_n\}$. Let us consider sets $\Omega_k = \{\boldsymbol{x} \in \Omega \mid \varphi\left(f\left(\boldsymbol{x}\right)\right) = b_k\}$, and restrictions $f_k\colon \Omega_k \to \mathbb{Z}_+$ of f on sets Ω_k. Then $\sum_{k=1}^{n} U\left(f_k\right) + U\left(f \circ \varphi\right) = U(f)$.

The basic theorem

Theorem 1

Let a functional U on $\mathcal{F}$ obey Axioms 1-6. Then $\bar{U}$ is the Shannon entropy, i.e.
$\bar{U}(f) = -c\sum_{i\in\mathbb{Z}_+} p_f(i)\ln p_f(i)$
and
$U(f) = -cN\sum_{i\in\mathbb{Z}_+} p_f(i)\ln p_f(i)$,
where N is a number of pixels in the image f and $c > 0$.

Measuring informativity and uncertainty of sign-based representations.

Notation: $U(\tau)$ is an information measure of the sing-based representation τ, $\hat{U}(\tau)$ is an uncertainty measure of τ.
We will characterize these measures using the following axioms.

Axiom 7.

Let $\tau \in \mathcal{T}$ and $U_{max}(\tau) = \sup\{U(f) \mid f \in \mathcal{F}_\tau\}$, then

$$U(\tau) + \hat{U}(\tau) = U_{max}(\tau).$$

This axiom expresses the principle formulated by George Klir that consists in the following. If we have an object with some informativity and its representation. Then the sum of uncertainty and informativity of the representation is equal to the informativity of the object.

Measuring informativity and uncertainty of sign-based representations. Continuation

Axiom 8

Let $\tau_1 \subseteq \tau_2$ for $\tau_1, \tau_2 \in \mathcal{T}$. Then $\hat{U}(\tau_1) \geqslant \hat{U}(\tau_2)$.

Images $f_1, f_2 \in \mathcal{F}$ are said to be equivalent, if there is a strictly increasing bijection $\varphi\colon f_1(\Omega) \to f_2(\Omega)$ such that $f_2 = \varphi \circ f_1$.
By our assumption, the equivalent images contain the same information.
This leads to the following axiom.

Axiom 9

$\hat{U}(\tau) = 0$ if the relation $\tau \in \mathcal{T}$ is connected, in other words, any pair of elements $\omega_1, \omega_2 \in \Omega$ is comparable.

Measuring informativity and uncertainty of sign-based representations. Continuation

Axiom 10

Let $G_\tau = (\Omega, \tau)$ be a graph of the sign-based representation $\tau \in \mathcal{T}$ and let sets $\Omega_1, \ldots, \Omega_m$ be connectivity components of G_τ. Then $\sum_{k=1}^{m} U\left(\tau_{\Omega_k}\right) = U(\tau)$, where $\tau_{\Omega_k} = \tau \cap \Omega_k \times \Omega_k$ is a restriction of τ on the set Ω_k, $k = 1, \ldots, m$.

Let us notice that the sence of Axiom 10 consists in the following. The connectivity components of G_τ can be considered as parts of independent information, therefore, we can assume that the information of the whole sign-based representation must be equal to the sum of informativities of independent components.

Corollaries from axioms

Proposition 1

Let $\tau \in \mathcal{T}$, $\theta = \tau \cap \tau^{-1}$ and let τ^{θ} be the extension of τ on the set $V = \{v_1, ..., v_n\}$ of equivalence classes, generated by θ. Then

$$U_{max}(\tau) = -cN \sum_{i=1}^{n} p(i) \ln p(i),$$

where $p(i) = |v_i|/N$ and $N = |\Omega|$.

Corollaries from axioms

Proposition 2

Let us notations from Proposition 1 be used, and $\theta = \tau$, i.e. τ is an equivalence relation. Then

$$\hat{U}(\tau) = -cN \sum_{i=1}^{n} p(i) \ln p(i),$$

where $p(i) = |v_i|/N$ и $N = |\Omega|$.

Corollaries from axioms

Proposition 3

Let G_τ be a graph of $\tau \in \mathcal{T}$, andlet its connectivity components be determined by sets $\Omega_1, \dots, \Omega_m$, in addition, τ_{Ω_i}, $i = 1, \dots, m$ are connected relations. Then

$$\hat{U}(\tau) = -cN \sum_{i=1}^{m} p(i) \ln p(i),$$

where $p(i) = |\Omega_i|/N$ и $N = |\Omega|$.

Corollaries from axioms

Proposition 4

Let $\tau \in \mathcal{T}$ and let $\alpha \subseteq \tau \cup \tau^{-1}$ be an equivalence relation. Then $\hat{U}(\tau) \leqslant \hat{U}(\alpha)$.

Proposition 4 allows us to introduce the following upper estimate $\hat{U}_{up}$ for uncertainty measure $\hat{U}$. Let $Eq\left(\tau \cup \tau^{-1}\right)$ be the family of all equivalence relations, which are included to the relation $\tau \cup \tau^{-1}$. Then the functional $\hat{U}_{up}$ is defined as follows:

$$\hat{U}_{up}(\tau) = \min\left\{\hat{U}(\alpha) \mid \alpha \in Eq\left(\tau \cup \tau^{-1}\right)\right\},$$

and by Proposition 4 $\hat{U}(\tau) \leqslant \hat{U}_{up}(\tau)$ for all $\tau \in \mathcal{T}$.

The basic theorem

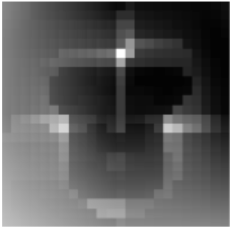
Theorem 2

The functional $\hat{U}_{up}(\tau)$, as a uncertainty measure, and the functional $U = U_{max} - \hat{U}_{up}$, as an information measure on the set $\mathcal{T}$ of sing-based representations obey Axioms 7-10.

Thus, according to Theorem 2 we can use as a measure of uncertainty for sign-based representations the functional $\hat{U}_{up}$. It is also important that the system of Axioms 7-10 is not contadictory.

Examples of recovering images by their sign-based representations using the entropy information measure.

initial image



$\varepsilon = 1$

$\varepsilon = 2$

$\varepsilon = 4$

$\varepsilon = 6$

$\varepsilon = \infty$