Residuated lattice of size ≤12

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Introduction

Ordered sets and lattices in computational intelligence

- **clustering:** hierarchies of clusters
  (e.g., formal concept analysis, ...)
- **uncertainty modeling:** scales in uncertainty theories
  (e.g., probability theory, Dempster-Shafer theory, fuzzy logics, ...)
  elements of scaled = **degrees** (of probability/plausibility/truth)

Typical choice: infinite real interval $[\mathbf{0}, \mathbf{1}]$

pros: infinite number of degrees is available

cons: problems that may be encountered:

- **non-linear scales** (e.g., information of two or more experts)
- **too many degrees** (Miller's $7{\pm}2$ phenomenon)
- **computational tractability** ($[0, 1]^X$ is always infinite)

Solution: infinite (linear) scale $\Longmapsto$ finite (non-linear) scale

Goal

Generate all n-element residuated lattices up to isomorphism

- **lattice generation**
  – methods for generating finite lattices up to n elements
- **quick isomorphism test**
  – heuristic test of non-isomorphism
- **generation of adjoint pairs**
  – generate all $\langle \otimes, \rightarrow \rangle$ for a given finite lattice
- **exploration of properties**
  – frequencies of properties
  – dependencies between properties
  – generated lattices can be used as counterexamples
  $\vdots$

Further Issues: hedges, independent negation-like connectives, ...

Results

Conference Presentations:

- Belohalvek, Vychodil: Counting finite residuated lattices.
  IFSA Congress 2007, Cancun, Mexico.
- Belohlavek, Vychodil: Scales behind computational intelligence: exploring properties of finite lattices.
  IEEE SSCI 2007 (FOCI), Honolulu, HI, USA.

Journal Paper:

- Belohlavek, Vychodil: Residuated Lattices of Size $\leq$12.
  Order **27**(2)(2010), 147–161.
- Extended version available at:
  `http://lattice.inf.upol.cz/order/reslat12.pdf`

On-line Database:

- `http:/lattice.inf.upol.cz/order/`

Preliminaries: Partially ordered sets

Partial order binary relation $\leq$ in U which is

reflexive: $a \leq a$,
antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$,
transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$.

Remark: $a < b$ is abbreviation for $a \leq b$ and $a \neq b$
if $a \leq b$ or $b \leq a$ then a, b are called **comparable**

Partially ordered set (poset) If $\leq$ is a partial order in U then $\mathbf{U} = \langle U, \leq \rangle$ is called a **partially ordered set.** $\mathbf{U} = \langle U, \leq \rangle$ is called a **totally (linearly) ordered set** / **chain** if each two elements from U are comparable.

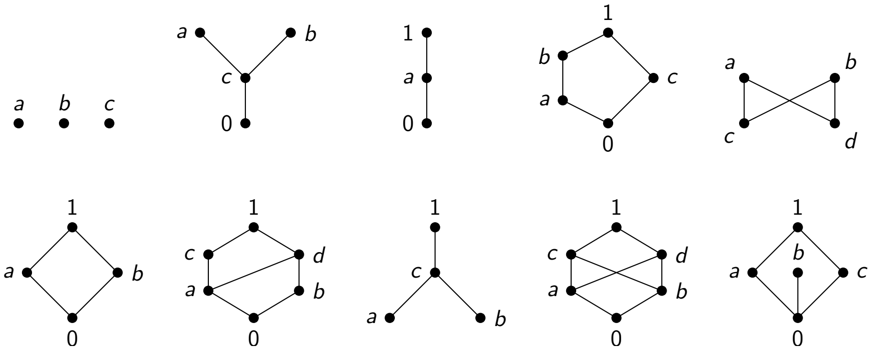
Special elements in posets $\langle U, \leq \rangle$ ... poset, $A \subseteq U$
$a \in A$ is **the least element of A** (w.r.t. $\leq$) if $a \leq b$ for each $b \in A$;
$a \in A$ is **the greatest element of A** (w.r.t. $\leq$) if $b \leq a$ for each $b \in A$;

Notation: 0 denotes the least element of U (if it exists)
1 denotes the greatest element of U (if it exists)

Preliminaries: Posets depicted by Hasse diagrams

Hasse diagrams

- **nodes**: elements $u \in U$
- **edges**: if $a < b$ and there is no $c \in U$ such that $a < c$ and $c < b$
  then there is edge between a and b, and a is put below b



Preliminaries: Cones, infima, suprema

Lower and upper cones

for $A \subseteq U$ define sets $\mathcal{L}(A), \mathcal{U}(A) \subseteq U$:

$$\mathcal{L}(A) = \{b \in U \mid b \leq a \text{ for each } a \in A\} \quad \ldots \quad \textbf{lower cone of } A$$
$$\mathcal{U}(A) = \{b \in U \mid a \leq b \text{ for each } a \in A\} \quad \ldots \quad \textbf{upper cone of } A$$

Upper and lower bounds

$b \in \mathcal{L}(A)$ is a lower bound of A (in $\langle U, \leq \rangle$),
$b \in \mathcal{U}(A)$ is an upper bound of A (in $\langle U, \leq \rangle$)

Infima and suprema

If $\mathcal{L}(A)$ has the greatest element then it is called **infimum of A** ... $\bigwedge A$
If $\mathcal{U}(A)$ has the least element then it is called **supremum of A** ... $\bigvee A$

By definition: $\bigwedge U = 0 = \bigvee \emptyset$ (if 0 exists in U)
$\bigwedge \emptyset = 1 = \bigvee U$ (if 1 exists in U)

Preliminaries: Lattices

Lattice ordered sets and lattices

$\leq$ (in U) is a **lattice order** if infimum and supremum exist for any $a, b \in U$

$\mathbf{L} = \langle L, \leq \rangle$ ... **lattice** (L + lattice order $\leq$ in L)

Finite lattice = lattice where L is finite

Notation: for $A = \{a, b\}$ we denote $\bigwedge A$ and $\bigvee A$ by $a \wedge b$ and $a \vee b$

Complete lattices

$\leq$ (in U) is a **complete lattice order** if inf. and sup. exist for any $A \subseteq U$

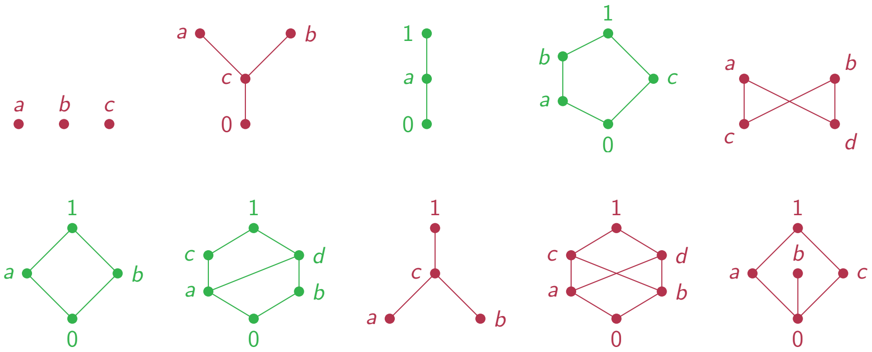
$\mathbf{L} = \langle L, \leq \rangle$ ... **complete lattice** (L + complete lattice order $\leq$ in L)

Examples: real unit interval $[0, 1]$ with its genuine ordering $\leq$ is complete
power set of X ordered by $\subseteq$ (set inclusion) is complete
interval $(0, 1]$ with its genuine ordering is not complete

Each finite lattice is complete:
for each subset $A = \{a_1, \ldots, a_n\} \subseteq L$: $\bigwedge A = a_1 \wedge a_2 \wedge \cdots \wedge a_n$

Example: Posets which are/are not lattices

- poset which is not lattice
- poset which is lattice



Indistinguishability of lattices

Motivation:

- some lattices are "the same" up to the "names of elements"
- finite isomorphic lattices have "the same Hasse diagram"

Formalization

Let $\mathbf{L}_1 = \langle L_1, \leq_1 \rangle$ and $\mathbf{L}_2 = \langle L_2, \leq_2 \rangle$ be lattices. A mapping $h\colon L_1 \to L_2$ is called a **lattice isomorphism** (between $\mathbf{L}_1$ and $\mathbf{L}_2$) if

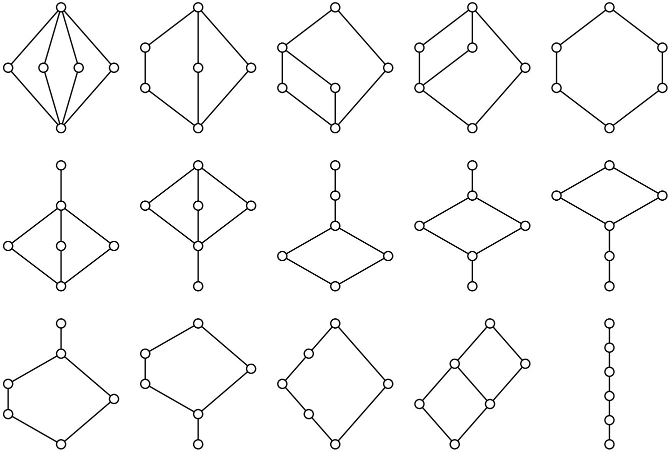
1. h is a bijection,
2. for each $a, b \in L_1$: $a \leq_1 b$ iff $h(a) \leq_2 h(b)$.

Lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are **isomorphic**, written $\mathbf{L}_1 \cong \mathbf{L}_2$, if there is a lattice isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$.

considering lattices "up to isomorphism" $=$ identifying all isomorphic ones

Observation: For $n \in \mathbb{N}$ there are only finitely many n-element lattices.

Preliminaries: Six-element lattices

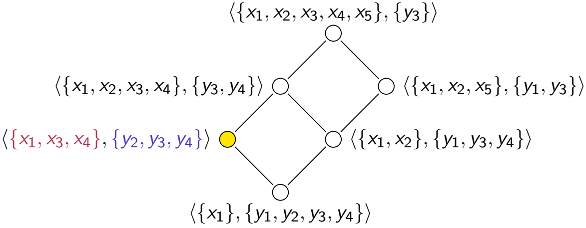


Example: Lattices as hierarchies of clusters

INPUT DATA: represented by a table

- table describing a relationship between **objects** and **attributes**
- **data tables with binary attributes**:
  "object has/does not have an attribute"

	y_1	y_2	y_3	y_4
x_1	×	×	×	×
x_2	×		×	×
x_3		×	×	×
x_4		×	×	×
x_5	×		×	



Conceptual clusters ... based on sharing of attributes

- input data table $\Longrightarrow$ **hierarchy of conceptual clusters**
- **Formal Concept Analysis (FCA)**
  R. Wille, 1980 (TU Darmstadt, Germany)

Representation of finite lattices

Finite lattice $\mathbf{L} = \langle \boldsymbol{L}, \boldsymbol{\leq} \rangle$

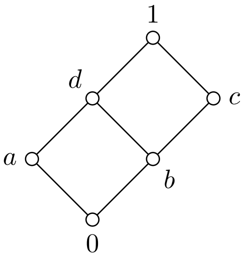
L ... finite support set $\{a_1, \ldots, a_n\}$

$\leq$ ... partial order

Adjacency matrices (tables)

- **rows** and **columns** ... elements from L
- **table entries** ... $\times$ or blank (or $1/0$)
- table entry given by row a and column b contains "$\times$" iff $a \leq b$

$\leq$	0	a	b	c	d	1
0	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
a		$\times$			$\times$	$\times$
b			$\times$	$\times$	$\times$	$\times$
c				$\times$		$\times$
d					$\times$	$\times$
1						$\times$



Adjacency tables in upper triangular form

Problem: $\mathbf{L} = \langle L, \leq \rangle \quad \Longmapsto \quad$ many adjacency tables

Solution based on:

Theorem

Fix L with $|L| = n$ and let $\preccurlyeq$ be a linear order in L. Then for each n-element lattice $\mathbf{L}' = \langle L', \leq' \rangle$ there is a lattice order $\leq$ in L such that

1. $\preccurlyeq$ extends $\leq$ and
2. $\mathbf{L}' = \langle L', \leq' \rangle$ is isomorphic to $\mathbf{L} = \langle L, \leq \rangle$.

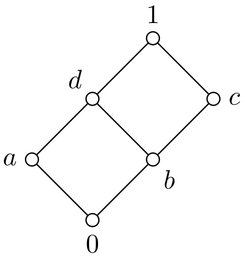
Each $\mathbf{L} = \langle L, \leq \rangle$ can be described by an upper-triangular adjacency table:

- find $\preccurlyeq$ described in previous Theorem,
- order columns and rows of adjacency table by $\preccurlyeq$.

Concise representation of finite lattices

Not all information in the upper triangular adjacency table is interesting

$\leq$	0	a	b	c	d	1
0	×	×	×	×	×	×
a		×			×	×
b			×	×	×	×
c				×		×
d					×	×
1						×



Mandatory records: $3 + \dfrac{n(n-5)}{2}$ $(n \geq 4)$

size of L	1	2	3	4	5	6	7	8	9	10	11
vector length	0	0	0	1	3	6	10	15	21	28	36
possible relations	2^0	2^0	2^0	2^1	2^3	2^6	2^{10}	2^{15}	2^{21}	2^{28}	2^{36}

Benefits of our representation

Efficient algorithms for computing $\wedge$ and $\vee$ in **L**

All essential algorithms are in $O(n)$

$L = \{a_1, \ldots, a_n\}$ and $\preccurlyeq$ such that $a_1 \preccurlyeq a_2 \preccurlyeq \cdots \preccurlyeq a_n$

INPUT: $a_i \preccurlyeq a_j$

procedure meet (a_i, a_j):
  if $a_i \leq a_j$**:**
    return a_i
  else:
    for k **from** $i - 1$ **downto** 1**:**
      if $a_k \leq a_i$ **and** $a_k \leq a_j$**:**
        return a_k

procedure join (a_i, a_j):
  if $a_i \leq a_j$**:**
    return a_j
  else:
    for k **from** $j + 1$ **upto** n**:**
      if $a_i \leq a_k$ **and** $a_j \leq a_k$**:**
        return a_k

The idea of generating all n-element lattices

(1) Begin with adjacency table filled with zeros.

(2) Check if the current table represents a lattice order.
If it represents a lattice order $\leq$, go to step (3).
Otherwise, go to step (4).

(3) Check if $\leq$ (lattice order represented by the current table) is **isomorphic** to a lattice which has already been generated.
   - If $\leq$ is not isomorphic to any of the generated lattices, add $\leq$ to the set of generated lattices and go to step (4).
   - If $\leq$ equals $\leq'$ and $\leq'$ was previously generated, then end this branch of recursion.
   - Otherwise, go to step (4).

(4) Loop over all bits of the binary vector $\leq$ which equal 0:
   - Make a copy $\leq'$ of $\leq$ (copy of adjacency tables).
   - Set to 1 the current bit in $\leq'$ which equals 0.
   - Make transitive closure of $\leq'$.
   - Recursively call (2) for $\leq'$.

Towards the heuristic test of non-isomorphism

Characteristics of lattice elements

Take $\mathbf{L} = \langle L, \leq \rangle$ and a linear order $\preccurlyeq$ extending $\leq$. Define $\mathcal{P}(L)$ by

$$\mathcal{P}(L) = \{\langle a, b\rangle \in \preccurlyeq \mid a \neq 0 \text{ and } b \neq 1 \text{ and } a \neq b\}\,.$$

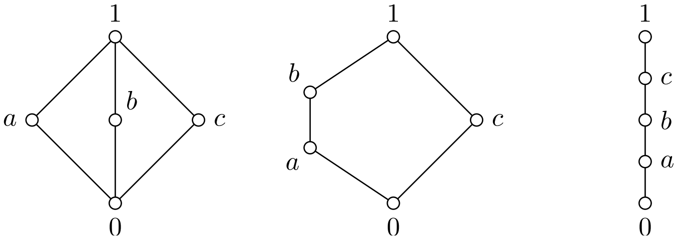
$\mathcal{P}(L)$ represents the non-trivial part of the upper triangular adjacency table

For each $a \in L$ we define:

$$\begin{aligned}
v_1(a) &= |\mathcal{L}(\{a\})| = |\{b \in L \mid b \leq a\}|\,,\\
v_2(a) &= |\mathcal{U}(\{a\})| = |\{b \in L \mid a \leq b\}|\,,\\
v_3(a) &= |\{\langle b, c\rangle \in \mathcal{P}(L) \mid a = b \wedge c\}|\,,\\
v_4(a) &= |\{\langle b, c\rangle \in \mathcal{P}(L) \mid a = b \vee c\}|\,.
\end{aligned}$$

$v_1(a)$ = number of elements which are lower than or equal to a
$v_2(a)$ = number of elements which are greater than or equal to a
$v_3(a)$ = number of non-trivial pairs of elements whose infimum is a
$v_4(a)$ = number of non-trivial pairs of elements whose supremum is a

Example: Characteristics of selected lattices



$\mathbf{L}_{5,1}$	0	a	b	c	1
v_1	1	2	2	2	5
v_2	5	2	2	2	1
v_3	3	0	0	0	0
v_4	0	0	0	0	3

$\mathbf{L}_{5,2}$	0	a	b	c	1
v_1	1	2	3	2	5
v_2	5	3	2	2	1
v_3	2	1	0	0	0
v_4	0	0	1	0	2

$\mathbf{L}_{5,5}$	0	a	b	c	1
v_1	1	2	3	4	5
v_2	5	4	3	2	1
v_3	0	2	1	0	0
v_4	0	0	1	2	0

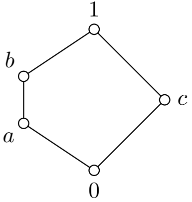
Characteristic vectors

Integers $v_1(a), \ldots, v_4(a)$**:** represent selected properties of $a \in L$ given by the "relative position" of $a \in L$ in the lattice.

For each $a \in L$ define a tuple: $v(a) = \langle v_1(a), v_2(a), v_3(a), v_4(a) \rangle$

Lexical order on elements of lattices: $a \leqslant_{\text{lex}} b$ iff either $v(a) = v(b)$ or there is $i \in \{1, \ldots, 4\}$ s.t., for each $j < i$, $v_j(a) = v_j(b)$ and $v_i(a) < v_i(b)$.

A **characteristic vector of $\mathbf{L} = \langle L, \leq \rangle$** is a vector of integers given by concatenation of vectors $v(a)$ $(a \in L)$ listed in the lexical order $\leqslant_{\text{lex}}$.



$\mathbf{L}_{5,2}$	0	a	b	c	1
v_1	1	2	3	2	5
v_2	5	3	2	2	1
v_3	2	1	0	0	0
v_4	0	0	1	0	2

$\Longrightarrow$

$\mathbf{L}_{5,2}$	0	c	a	b	1
v_1	1	2	2	3	5
v_2	5	2	3	2	1
v_3	2	0	1	0	0
v_4	0	0	0	1	2

characteristic vector: $\langle 1, 5, 2, 0, 2, 2, 0, 0, 2, 3, 1, 0, 3, 2, 0, 1, 5, 1, 0, 2 \rangle$

Heuristic test of non-isomorphism

The heuristic test is based on:

Theorem

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be finite lattices, $h\colon L_1 \to L_2$ be a lattice isomorphism. Then, for each $a \in L_1$, we get $v^{\mathbf{L}_1}(a) = v^{\mathbf{L}_2}(h(a))$. As a consequence, the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ have the same characteristic vectors.

Heuristic test can "fail":

some non-isomorphic lattices have the same characteristic vectors

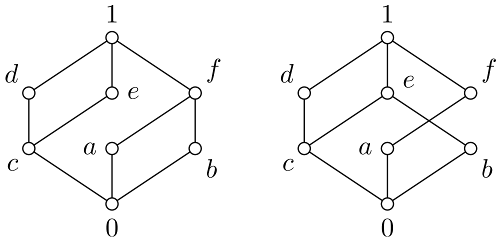
Complete test of isomorphism of $\mathbf{L_1}$ and $\mathbf{L_2}$

1. compute char. vectors of $\mathbf{L}_1$ and $\mathbf{L}_2$ (can be done in $O(n^3)$)
   if the vectors do not agree, answer **"not isomorphic"**, else:
2. for each bijection $h\colon L_1 \to L_2$, s.t. $v^{\mathbf{L}_1}(a) = v^{\mathbf{L}_2}(h(a))$ ($a \in L_1$):
   if h is isomorphism then answer **"isomorphic"**
3. answer **"not isomorphic"**

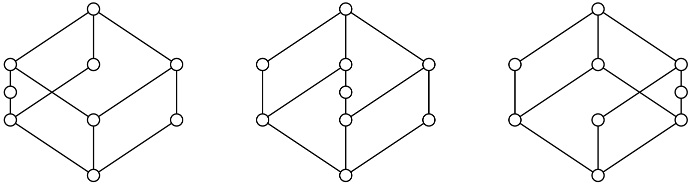
Example: When the heuristic test fails

The heuristic test **never fails for L with $|L| \leq 7$**.

The smallest lattices on which the test fails:



Three non-isomorphic lattices sharing the same characteristic vector:



Probability of failure

Order of a characteristic vector c = the number of pairwise non-isomorphic lattices whose characteristic vector is exactly c

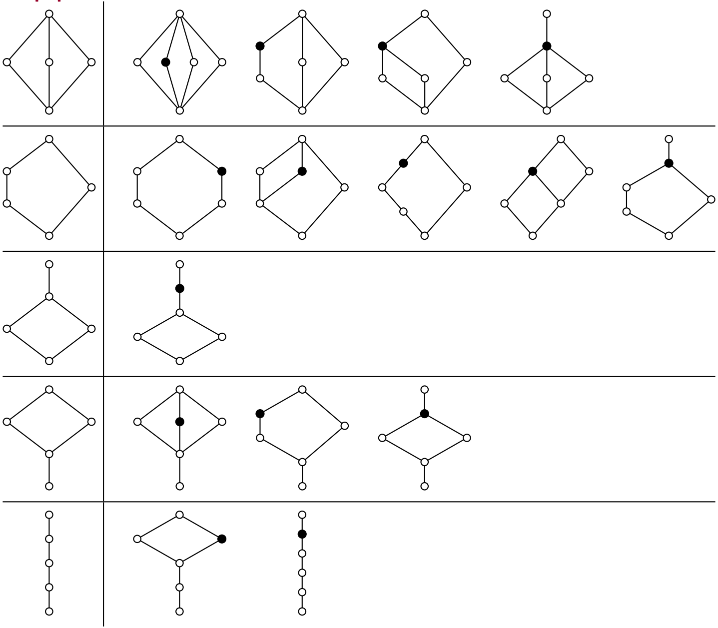
Numbers of characteristic vectors of given orders

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	2	5	15	53	220	1049	5682	34502
2	0	0	0	0	0	0	0	1	13	125	1159
3	0	0	0	0	0	0	0	0	1	18	212
4	0	0	0	0	0	0	0	0	0	2	28
5	0	0	0	0	0	0	0	0	0	0	6
6	0	0	0	0	0	0	0	0	0	0	4

Probability of failure of the heuristic test

size of L	8	9	10	11
probability	$4.06 \cdot 10^{-5}$	$2.75 \cdot 10^{-5}$	$1.06 \cdot 10^{-5}$	$2.94 \cdot 10^{-6}$

Incremental Approach



Residuated lattices

(Complete) residuated lattice – basic structure of truth degrees

$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, where
- $\langle L, \wedge, \vee, 0, 1\rangle$ ... (complete) lattice,
- $\langle L, \otimes, 1\rangle$ ... commutative monoid,
- $\langle \otimes, \rightarrow\rangle$ ... adjoint pair ($a \otimes b \leq c$ iff $a \leq b \rightarrow c$).

Structures on $[0, 1]$ (based on t-norms)

$\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ given by left-continuous (continuous) $\otimes$.

Łukasiewicz:

$$a \otimes b = \max(a + b - 1, 0),$$
$$a \rightarrow b = \min(1 - a + b, 1),$$

Gödel (minimum):

$$a \otimes b = \min(a, b),$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases}$$

Finite structures of truth degrees

$L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ... finite subset of $[0, 1]$

finite Łukasiewicz chain / Gödel chain ... restrictions of $\otimes, \rightarrow$ on finite L

Representation of finite residuated lattices

Finite residuated lattice

- **finite lattice** (as before)
- **multiplication** (two-dimensional table)
- **residuum** – superfluous, because:

Theorem (Equivalent formulations of adjointness.)

Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a finite lattice, $\langle L, \otimes, 1\rangle$ be a commutative monoid s. t. $\otimes$ is monotone w.r.t. $\leq$. Then the following are equivalent:

1. there exists (unique) $\rightarrow$ satisfying adjointness w.r.t. $\otimes$;
2. for each $a, b, c \in L$: $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$;
3. $a \rightarrow b = \bigvee\{c \in L \mid a \otimes c \leq b\}$ satisfies adjointness w.r.t. $\otimes$.

Conclusion: lattice part and $\otimes$ are sufficient

Representation of multiplication

Multiplications are encoded in two-dimensional tables

$\otimes$	0	a_1	a_2	$\cdots$	a_{n-2}	1
0	0	0	0	$\cdots$	0	0
a_1	0					a_1
a_2	0					a_2
$\vdots$	$\vdots$					$\vdots$
a_{n-2}	0					a_{n-2}
1	0	a_1	a_2	$\cdots$	a_{n-2}	1

Values of $a \otimes b$ can be bounded from below and above

Theorem (Bounds for the value of $a \otimes b$.)

Let **L** be a residuated lattice. Then, for each $a, b \in L$,

1. $a \otimes b \leq a \wedge b$;
2. $\bigvee\{c \otimes d \mid c, d \in L$ such that $c \leq a$ and $d \leq b\} \leq a \otimes b$.

Generating all n-element residuated lattices

Algorithm

For each n-element lattice:

succesively generate all possible tables for $\otimes$

Entries of tables for $\otimes$ are generated so that:

- each table entry respects the bounds,
- elements generated so far satisfy: **associativity**, **monotony** +

$$a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c).$$

	1	2	3	4	5	6	7	8	9	10	11	12
ℓ	1	1	1	2	5	15	53	222	1078	5994	37622	262776
residuated ℓ	1	1	2	7	26	129	723	4712	34698	290565	2779183	30653419
lin. res. ℓ	1	1	2	6	22	94	451	2386	13775	86417	590489	4446029
res. ℓ reducts	1	1	1	2	3	7	18	61	239	1125	6138	38165

Automorphism Issues

Automprphic Residuated Lattices

- $\mathbf{L}_1 \cong \mathbf{L}_2$:
- there is a bijective map $h\colon L_1 \to L_2$ such that
  - $a \leq_1 b$ iff $h(a) \leq_2 h(b)$ (lattice isomorphism),
  - $h(a \otimes_1 b) = h(a) \otimes_2 h(b)$.

Issues with Previous Method

- For $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$, it generates all possible pairs $\langle \otimes, \rightarrow \rangle$ of adjoint operations, and
- there can be different $\langle \otimes_1, \rightarrow_1 \rangle$ and $\langle \otimes_2, \rightarrow_2 \rangle$ so that for $\mathbf{L}_1 = \langle L, \wedge, \vee, \otimes_1, \rightarrow_1, 0, 1 \rangle$ and $\mathbf{L}_2 = \langle L, \wedge, \vee, \otimes_2, \rightarrow_2, 0, 1 \rangle$ we get $\mathbf{L}_1 \cong \mathbf{L}_2$.

Removing Automorphic Copies

- particular total order is defined on all possible $\otimes$,
- $\otimes$ is stored iff it is least with respect to the total order,
- relies on generating all automorphisms $h\colon L \to L$ (with help of characteristic vectors).

Numbers of generated lattices

Numbers of non-isomorphic lattices + their properties

	1	2	3	4	5	6	7	8	9	10	11	12
all lattices	1	1	1	2	5	15	53	222	1078	5994	37622	262776
modular	1	1	1	2	4	8	16	34	72	157	343	766
distributive	1	1	1	2	3	5	8	15	26	47	82	151
complemented	1	1	0	1	2	6	18	71	307	1594	9446	63461
Boolean	1	1	0	1	0	0	0	1	0	0	0	0
relatively cmpl.	1	1	0	1	1	1	1	2	2	4	6	13
pseudo-cmpl.	1	1	1	2	4	10	29	99	391	1775	9214	54151
rel. pseudo-cmpl.	1	1	1	2	3	5	8	15	26	47	82	151

Numbers of generated residuated lattice

Numbers of residuated lattices with given properties

	1	2	3	4	5	6	7	8	9	10	11	12
res. ℓ	1	1	2	7	26	129	723	4712	34698	290565	2779183	30653419
modular	1	1	2	7	26	125	660	3923	25445	180113	1389782	11798582
distributive	1	1	2	7	26	124	645	3792	24268	169553	1290956	10823436
Π1	1	1	1	4	9	46	240	1610	12679	118052	1280764	16074272
prelinear	1	1	2	7	23	99	464	2453	14087	88188	601205	4516962
Π2	1	1	1	3	8	30	143	794	5090	37036	306456	2897889
strict	1	1	1	3	7	27	129	726	4713	34705	290565	2779212
wnm	1	1	2	5	11	30	78	238	771	2908	12812	67467
divisible	1	1	2	5	10	23	49	111	244	545	1203	2697
involutive	1	1	1	3	3	12	15	70	112	493	980	4325
idempotent	1	1	1	2	3	5	8	15	26	47	82	151

Numbers of important algebras

Numbers of residuated lattices with given properties

	1	2	3	4	5	6	7	8	9	10	11	12
all res. ℓ	1	1	2	7	26	129	723	4712	34698	290565	2779183	30653419
MTL	1	1	2	7	23	99	464	2453	14087	88188	601205	4516962
SMTL	1	1	1	3	7	24	99	467	2454	14094	88188	601231
WNM	1	1	2	5	9	21	40	90	180	378	757	1584
BL	1	1	2	5	9	20	38	81	160	326	643	1314
SBL	1	1	1	3	5	10	20	41	82	165	326	655
IMTL	1	1	1	3	3	8	12	35	61	167	333	971
Heyting al.	1	1	1	2	3	5	8	15	26	47	82	151
G	1	1	1	2	2	3	3	5	6	8	8	12
NM	1	1	1	2	1	2	1	4	3	3	2	6
MV	1	1	1	2	1	2	1	3	2	2	1	4
Π	1	1	0	1	0	0	0	1	0	0	0	0
ΠMTL	1	1	0	1	0	0	0	1	0	0	0	0

Residuated lattices sharing the same properties

	MTL	SMTL	WNM	BL	SBL	IMTL	G	NM	MV	Π	ΠMTL
28539974											
4511103	$\times$										
705260	$\times$	$\times$									
2954	$\times$		$\times$								
1556	$\times$					$\times$					
1258	$\times$	$\times$		$\times$	$\times$						
1234	$\times$			$\times$							
48	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$				
39	$\times$		$\times$	$\times$							
19	$\times$		$\times$			$\times$		$\times$			
13	$\times$			$\times$		$\times$			$\times$		
4	$\times$		$\times$	$\times$		$\times$		$\times$	$\times$		
4	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Distribution of lattices according to their dimensions

Numbers of 12-element lattices of given dimensions

- row = **height** of lattices (length of the longest max. chain)
- column = **width** of lattices (length of the longest max. antichain)

	1	2	3	4	5	6	7	8	9	10
3										1
4					99	395	288	98	17	
5				3847	14418	9536	2115	176		
6			3531	37813	43394	12050	952			
7		87	15501	48261	23595	2507				
8		666	14735	17380	3117					
9		849	4704	1792						
10		350	456							
11		45								
12	1									

Distribution of lattices according to their dimensions

Numbers of 12-element residuated lattices of given dimensions

- row = **height** of lattices (length of the longest max. chain)
- column = **width** of lattices (length of the longest max. antichain)

	1	2	3	4	5	6	7	8	9
4									1
5				3	127	165	88	48	
6			240	9383	22627	9638	1335		
7		236	99088	332299	161275	18546			
8		121970	1363290	1009364	142551				
9		1732870	3563657	733266					
10		6007716	2709365						
11		8168242							
12	4446029								

Distribution of lattices according to their dimensions

Numbers of lattice reducts of 12-element residuated lattices

- row = **height** of lattices (length of the longest max. chain)
- column = **width** of lattices (length of the longest max. antichain)

	1	2	3	4	5	6	7	8	9
4									1
5				2	123	159	72	15	
6			92	2215	3295	1139	126		
7		11	2362	8498	4397	518			
8		241	4549	5377	973				
9		455	2183	805					
10		239	280						
11		37							
12	1								

Conclusions and future work

Future work

- incremental algorithms
  - has been done for finite lattices
  - open problem for finite residuated lattices
- exploration of further properties of the generated lattices
- improvements of heuristic tests of non-isomorphism
- query languages over finite structures (based on residuated lattices)
- combination of our approach with other methods:
  Heitzig J., Reinhold J.: Counting Finite Lattices.
  Algebra Universalis **48**(1)2002, 43–53
- generation of lattices extended by additional operations
  (e.g., residuated lattices, lattices with hedges, ...)