Triadic concept analysis of data with fuzzy attributes

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Problem setting & motivation

- Growing interest in analyzing three-way (multi-way) data
  Kolda T. G., Bader B. W.: Tensor decompositions and applications. *SIAM Review* **51**(3)(2009), 455–500.
- Triadic concept analysis (TCA) — particular method of analysis of three-way relational data, Rudolf Wille 1995, inspired by Peirces categories
- Main motivation is to develop foundations for factor analysis of three-way relational data

Contribution:

- Fundamental notions and results for TCA of data with fuzzy attributes
- Proof of the generalization of basic theorem of TCA (Wille, 1995)

Triadic concept analysis (TCA)

z_1	y_1	y_2	y_3	y_4
x_1	1	1	1	1
x_2	1	0	1	1
x_3	0	1	1	1
x_4	0	1	1	1
x_5	1	0	0	0

z_2	y_1	y_2	y_3	y_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	1	1	1
x_4	1	0	0	1
x_5	1	1	1	0

z_3	y_1	y_2	y_3	y_4
x_1	1	1	1	0
x_2	0	1	1	1
x_3	0	1	1	1
x_4	1	1	0	1
x_5	1	0	0	1

Formally:

$X = \{x_1, x_2, \dots\}$ set (of **objects**)

$Y = \{y_1, y_2, \dots\}$ set (of **attributes**)

$Z = \{z_1, z_2, \dots\}$ set (of **conditions**)

$I \subseteq X \times Y \times Z$ ternary relation (**to have under**)

$\langle x, y, z \rangle \in I$ object x has attribute y under condition z

Triadic context

$\mathbf{K} = \langle X_1, X_2, X_3, I \rangle$, $\{i, j, k\} = \{1, 2, 3\}$

Induced contexts:
For $C_k \subseteq X_k$ $\mathbf{K}$ induces dyadic formal contexts $\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k} \rangle$ such that

$$\langle x_i, x_j \rangle \in I^{ij}_{C_k} \text{ iff for each } x_k \in C_k : x_i, x_j, x_k \text{ are related by } I.$$

The concept forming operators induced by $K^{ij}_{C_k}$ are denoted by ${}^{(ijC_k)}$.

Triadic concept:
a triple $\langle A_1, A_2, A_3 \rangle$, $A_i \subseteq X_i$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have

$$A_i = A_j^{(ijC_k)}.$$

A_1, A_2, A_3 are called the **extent**, **intent**, and **modus**. A set of all triadic concepts $\mathcal{T}(\mathbf{K})$ resulting from $\mathbf{K}$ is called **concept trilattice**.

TCA of data with fuzzy attributes

Input data

We want the **to have under** relation to be graded rather than yes-or-no (the grades are taken from a residuated lattice)

Example:

objects ... products (eg. food products)
attributes ... features of products (eg. taste)
conditions ... customers (eg. participating in a survey)
$\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ interpreted as { "very bad", "bad", "neutral", "good", "excelent" }

Object x has attribute y under condition z to the degree $\frac{3}{4}$ is interpreted as the customer z considering the product x as having "good" z feature.

Triadic fuzzy context:
$\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$, where $I \subseteq L^{X_1\times X_2\times X_3}$ is a fuzzy relation.

We denote $I(x, y, z)$ also by $I\{x, y, z\}$ or $I\{x, z, y\}$ or $I\{z, x, y\}$

Concept forming operators

For $C_k \in L^{X_k}$ $\mathbf{K}$ induces a dyadic fuzzy context $\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k} \rangle$ with $I^{ij}_{C_k} \in L^{X_i \times X_j}$ defined by

$$I^{ij}_{C_k}(x_i, x_j) = \bigwedge_{x_k \in X_k} (C_k(x_k) \to I\{x_i, x_j, x_k\})$$

The concept-forming operators induced by $\mathbf{K}^{ij}_{C_k}$ are denoted by ${}^{(i,j,C_k)}$.

For a fuzzy set C_i in X_i, we define a fuzzy set $C_i^{(i,j,C_k)}$ in X_j by

$$C_i^{(i,j,C_k)}(x_j) \;=\; \bigwedge_{x_i \in X_i} C_i(x_i) \to I^{ij}_{C_k}\{x_i, x_j\}.$$

Triadic fuzzy concept

We want concept to apply to different objects (attributes, conditions) to possibly different degrees.

A **triadic fuzzy concept** of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle A_1, A_2, A_3\rangle$ consisting of fuzzy sets $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have

$$A_i = A_j^{(i,j,A_k)},\ A_j = A_k^{(j,k,A_i)},\ \text{and}\ A_k = A_i^{(k,i,A_j)}.$$

A_1, A_2, and A_3 are called the **extent**, **intent**, and **modus** of $\langle A_1, A_2, A_3\rangle$.

The set of all triadic concepts of $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the **concept trilattice of $\mathbf{K}$.**

Geometrical interpretation

Triadic fuzzy concepts = maximal cubical patterns in data

Theorem

(a) *If* $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K})$ *then* $A_1 \otimes A_2 \otimes A_3 \subseteq I$. *Moreover,* $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K})$ *is maximal with respect to pointwise set inclusion, i.e. there does not exist a different* $\langle B_1, B_2, B_3 \rangle \in \mathcal{T}(\mathbf{K})$ *for which* $A_i \subseteq B_i$.

(b) *If* $A_1 \otimes A_2 \otimes A_3 \subseteq I$ *then there is* $\langle B_1, B_2, B_3 \rangle \in \mathcal{T}(\mathbf{K})$ *such that* $A_i \subseteq B_i$ *for* $i = 1, 2, 3$.

Consequences important for decomposition of three-way matrices:

- the existence of decomposition using triadic concepts as factors is ensured by

$$I(x_1, x_2, x_3) = \bigvee_{\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\langle X_1, X_2, X_3, I \rangle)} A_1(x_1) \otimes A_2(x_2) \otimes A_3(x_3)$$

- maximality of triadic concepts implies optimality of such decomposition

Structure of triadic concepts - preliminaries

Triordered set:
$\mathcal{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3 \rangle$ where for $\{i, j, k\} = \{1, 2, 3\}$, $\lesssim_i$ is a quasiorder (reflexive and transitive relation) such that for all $w, v \in V$

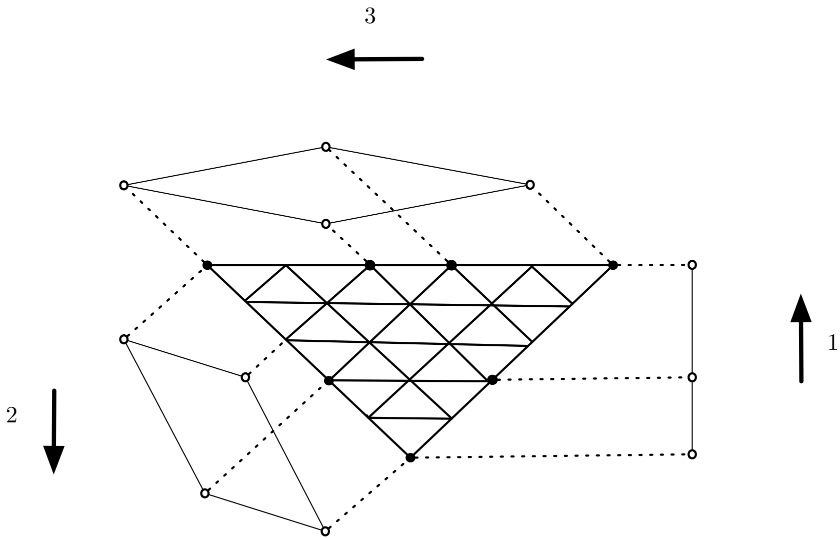
(i) $v \lesssim_i w$ and $v \lesssim_j w$ implies $w \lesssim_k v$ (antiordinal dependency)

(ii) $\sim_i \cap \sim_j \cap \sim_k$ (with $\sim_i = \lesssim_i \cap \gtrsim_i$) is an identity relation.

Remarks:

- $\sim_i$ is an equivalence
- $\sim_i \cap \sim_j$ is an identity relation on V.
- $\lesssim_i$ induces a partial order on the set $V/\sim_i$ of equivalence classes of $\sim_i$.

Diagrams of triordered sets



Trilattice

Element $v \in V$ is:

- ik-**bound** of $\langle V_i, V_k \rangle$: $x \lesssim_i v$ for all $x \in V_i$ and $x \lesssim_k v$ for all $x \in V_k$.
- ik-**limit** of $\langle V_i, V_k \rangle$: $u \lesssim_j v$ for all ik-bounds u of $\langle V_i, V_k \rangle$

In every triordered set there is at most one ik-limit v of $\langle V_i, V_k \rangle$ such that $u \lesssim_i v$ (equivalently $v \lesssim_k u$) for all ik-limits u of $\langle V_i, V_k \rangle$. If such exists, we call it ik-**join** of $\langle V_i, V_k \rangle$.

A triordered set $\mathcal{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3 \rangle$ in which ik-joins exist for all pairs of subsets of V is a **complete trilattice**.

Structure of $\mathcal{T}(\mathbf{K})$

Consider the following relations on $\mathcal{T}(\mathbf{K})$, for $i = 1, 2, 3$:

$$\langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle \quad \text{iff} \quad A_i \subseteq B_i,$$
$$\langle A_1, A_2, A_3\rangle \eqsim_i \langle B_1, B_2, B_3\rangle \quad \text{iff} \quad A_i = B_i.$$

$\lesssim_i$ and $\eqsim_i$ are a quasiorder and an equivalence on $\mathcal{T}(\mathbf{K})$.

Theorem ($\mathcal{T}(\mathbf{K})$ is a triordered set)

Let $\{i, j, k\} = \{1, 2, 3\}$. Then for all triadic fuzzy concepts $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ from $\mathcal{T}(\mathbf{K})$ holds,

$$\text{if } \langle A_1, A_2, A_2\rangle \lesssim_i \langle B_1, B_2, B_3\rangle \text{ and } \langle A_1, A_2, A_2\rangle \lesssim_j \langle B_1, B_2, B_3\rangle \text{ then } \langle B_1, B_2, B_3\rangle \lesssim_k \langle A_1, A_2, A_3\rangle.$$

Furthemore, $\eqsim_i \cap \eqsim_j$ is the identity on $\mathcal{T}(\mathbf{K})$.

Computation of a triadic fuzzy concept

Theorem

For $C_i \in L^{X_i}, C_k \in L^{X_k}$ *with* $\{i, j, k\} = \{1, 2, 3\}$, *let*

$$A_j = C_i^{(i,j,C_k)}, A_i = A_j^{(i,j,C_k)}, A_k = A_i^{(i,k,A_j)}.$$

Then $\langle A_1, A_2, A_3 \rangle$ *is a triadic fuzzy concept denoted as* $\mathfrak{b}_{ik}(C_i, C_k)$.

Moreover, $\langle A_1, A_2, A_3 \rangle$ *has the smallest* k*-th component among all triadic fuzzy concepts* $\langle B_1, B_2, B_3 \rangle$ *with the greatest* j*-th component satisfying* $X_i \subseteq B_i$ *and* $X_k \subseteq B_k$.

Basic theorem

Generalizes the first part of the basic theorem of ordinary triadic concept analysis.

Theorem (Basic theorem, part 1)

Let $\mathbf{K} = (X_1, X_2, X_3, I)$ *be a triadic fuzzy context. Then* $\mathcal{T}(\mathbf{K})$ *is a complete trilattice of* $\mathbf{K}$ *for which the* ik*-joins are defined as follows:*

$$\begin{aligned} \nabla_{ik}(\mathcal{X}_i, \mathcal{X}_j) \;&=\; \mathfrak{b}_{ik}\Big(\bigcup\{A_i | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\}, \\ &\qquad \bigcup\{A_k | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\}\Big). \end{aligned}$$

Structure of Structure of $\mathcal{T}(\mathbf{K})$

An **order filter** in the quasiordered set $\langle V, \lesssim_i \rangle$ is a subset $F \subseteq V$ for which $v \in F$ whenever $u \in F$ and $u \lesssim_i v$ for every $u, v \in V$. The set of all order filters of $\langle V, \lesssim_i \rangle$ is denoted by $\mathcal{F}_i(\mathbf{V})$.

A **principal filter** of $\langle V, \lesssim_i \rangle$ generated by $v \in V$ is the order filter $[v)_i = \{u \in V \mid v \lesssim_i u\}$.

A subset $\mathcal{X} \subseteq \mathcal{F}_i(\mathbf{V})$ is called ***i*-dense** with respect to $\mathbf{V}$ if each principal filter of $\langle V, \lesssim_i \rangle$ is the intersection of some order filters from $\mathcal{X}$.

Basic theorem

Theorem (Basic theorem, part 2)

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ *be a triadic fuzzy context. A complete trilattice* $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ *is isomorphic to* $\mathcal{T}(\mathbf{K})$ *if and only if there are mappings* $\tilde{\kappa}_i : X_i \times L \to \mathcal{F}_i(\mathbf{V})$, $i = 1, 2, 3$, *such that*

(i) $\tilde{\kappa}_i(X_i \times L)$ *is* i*-dense with respect to* $\mathbf{V}$,

(ii) $A_1 \otimes A_2 \otimes A_3 \subseteq I$ *iff* $\bigcap_{i=1}^{3} \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$, *for every* $A_i \in L^{X_i}$,

(iii) $a \leq b$ *implies* $\tilde{\kappa}_i(x_i, b) \subseteq \tilde{\kappa}_i(x_i, a)$ *for every* $a, b \in L$, $x_i \in X_i$, $i = 1, 2, 3$.

Open problem: can (iii) be proved from (i) and (ii)?

Crisp reprezentation

Theorem (crisp representation)

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ *be a fuzzy triadic context and* $\mathbf{K}_{crisp} = \langle X_1 \times L, X_2 \times L, X_3 \times L, I_{crisp}\rangle$ *with* I_{crisp} *defined by*

$$((x_1, a), (x_2, b), (x_3, c)) \in I_{crisp} \text{ iff } a \otimes b \otimes c \leq I(x_1, x_2, x_3)$$

be a triadic context. Then $\mathcal{T}(\mathbf{K})$ *is isomorphic to* $\mathcal{T}(\mathbf{K}_{crisp})$.

Remarks:

- isomorphism: $\varphi(\langle A_1, A_2, A_3\rangle) = \langle \lfloor A_1 \rfloor, \lfloor A_2 \rfloor, \lfloor A_3 \rfloor\rangle$, where for $A \in L^X$ we define $\lfloor A \rfloor = \{(x, a) \mid x \in X, a \in L, A(x) \geq a\}$,
- allows us to use crisp algorithms to compute triadic fuzzy concepts,
- can be utilized in alternative proof of basic theorem.

Example - Input data

- restaurant survey (objects = dishes, attributes = features, conditions = customers)
- three element scale of truth: $\{0, \frac{1}{2}, 1\}$ representing "bad", "neutral" and "excellent"

Fry	t	a	l	p
steak	1	1	1	0
salad	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
veget.	0	0	$\frac{1}{2}$	1
wings	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Bender	t	a	l	p
steak	1	1	1	$\frac{1}{2}$
salad	0	0	$\frac{1}{2}$	0
veget.	0	0	0	0
wings	1	1	$\frac{1}{2}$	1

Leela	t	a	l	p
steak	$\frac{1}{2}$	0	$\frac{1}{2}$	0
salad	1	1	1	1
veget.	$\frac{1}{2}$	0	$\frac{1}{2}$	0
wings	0	0	0	$\frac{1}{2}$

Zoidberg	t	a	l	p
steak	1	1	1	0
salad	1	1	1	$\frac{1}{2}$
veget.	1	1	1	0
wings	1	1	1	$\frac{1}{2}$

Example - Interesting concepts

	1	2	3	4	5
steak	1	0	1	1	1
salad	0	1	1	0	1
veget.	0	0	1	0	1
wings	1	0	1	$\frac{1}{2}$	1
taste	1	1	1	1	1
aroma	1	1	1	$\frac{1}{2}$	1
look	$\frac{1}{2}$	1	1	1	1
price	0	$\frac{1}{2}$	0	0	1
Fry	1	0	0	1	0
Bender	1	0	0	1	0
Leila	0	1	0	0	0
Zoidberg	1	1	1	1	0

- Concept no. 1 describes customers who like meat dishes for their taste and aroma.
- Concept no. 2 represents customers who like cheese salad for its excellent taste, aroma and look, and partly for its price
- Concept no. 3 can be interpreted as "customers who have no preferences in food."

Conclusions

Related papers:

- Belohlavek R., Osicka P.: Triadic concept analysis of data with fuzzy attributes. *Proc. of The 2010 IEEE International Conference on Granular Computing (GrC 2010)*, 2010, San José, USA
- Belohlavek R., Osicka P.: Triadic concept lattices of data with graded attributes. *International Journal of General Systems* (submitted)

Related results:

- Belohlavek R., Osicka P., Vychodil V.: Factorizing three-way ordinal data using triadic formal concepts, *FQAS 2011*, Ghent, Belgium
- Konecny J., Osicka P.: General approach to triadic concept analysis. *The 7th International Conference on Concept Lattices and their Applications, 2010*, Sevilla, Spain

Future research

- triadic galois connections and their properties.
- structure of $\mathcal{T}(\mathbf{K})$ w.r.t fuzzy quasiorders
- algorithms
- triadic atribute implications